International
Association
of Oil & Gas
Producers

Surveying and Positioning
Guidance Note Number 7, part 2

Coordinate Conversions and Transformations including Formulas

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Index

Preface

The EPSG Geodetic Parameter Dataset, abbreviated to the **EPSG Dataset**, is a repository of parameters required to:

- define a *coordinate reference system* (CRS) which ensures that coordinates describe position unambiguously.
- define transformations and conversions that allow coordinates to be changed from one CRS to another CRS. Transformations and conversions are collectively called *coordinate operations*.

The EPSG Dataset is maintained by the OGP Surveying and Positioning Committee's Geodetic Subcommittee. It conforms to ISO 19111 – *Spatial referencing by coordinates*. It is distributed in three ways:

- the **EPSG Registry**, in full the *EPSG Geodetic Parameter Registry*, a web-based delivery platform in which the data is held in GML using the CRS entities described in ISO 19136.
- the **EPSG Database**, in full *the EPSG Geodetic Parameter Database*, a relational database structure where the entities which form the components of CRSs and coordinate operations are in separate tables, distributed as an MS Access database;
- in a relational data model as **SQL scripts** which enable a user to create an Oracle, MySQL, PostgreSQL or other relational database and populate that database with the EPSG Dataset;

OGP Surveying and Positioning Guidance Note 7 is a multi-part document for users of the EPSG Dataset.

- *Part 0, Quick Start Guide*, gives a basic overview of the Dataset and its use.

- *Part 1, Using the Dataset*, sets out detailed information about the Dataset and its content, maintenance and terms of use.

- *Part 2, Formulas*, (this document), provides a detailed explanation of formulas necessary for executing coordinate conversions and transformations using the coordinate operation methods supported in the EPSG dataset. Geodetic parameters in the Dataset are consistent with these formulas.

- *Part 3, Registry Developer Guide*, is primarily intended to assist computer application developers who wish to use the API of the Registry to query and retrieve entities and attributes from the dataset.

- *Part 4, Database Developer Guide*, is primarily intended to assist computer application developers who wish to use the Database or its relational data model to query and retrieve entities and attributes from the dataset.

The complete text may be found at http://www.epsg.org/guides/docs/G7.html. The terms of use of the dataset are also available at http://www.epsg.org/CurrentDB.html.

In addition to these documents, the Registry user interface contains online help and the Database user interface includes context-sensitive help accessed by left-clicking on any label.

This Part 2 of the multipart Guidance Note is primarily intended to assist computer application developers in using the coordinate operation methods supported by the EPSG Database. It may also be useful to other users of the data.

A **coordinate system** is a set of mathematical rules for specifying how coordinates are to be assigned to points. It includes the definition of the coordinate axes, the units to be used and the geometry of the axes. The coordinate system is unrelated to the Earth. A **coordinate reference system** (CRS) is a coordinate system related to the Earth through a **datum**. Colloquially the term coordinate system has historically been used to mean coordinate reference system.

Coordinates may be changed from one coordinate reference system to another through the application of a **coordinate operation**. Two types of coordinate operation may be distinguished:

- **coordinate conversion**, where no change of datum is involved and the parameters are chosen and thus error free.
- **coordinate transformation**, where the target CRS is based on a different datum to the source CRS. Transformation parameters are empirically determined and thus subject to measurement errors.

A projected coordinate reference system is the result of the application of a **map projection** to a geographic coordinate reference system. A map projection is a type of coordinate conversion. It uses an identified method with specific formulas and a set of parameters specific to that coordinate conversion method.

Map projection methods are described in section 1 below. Other coordinate conversions and transformations are described in section 2.

Revision history:

Version	Date	Amendments
1	December 1993	First release – POSC Epicentre
10	May 1998	Additionally issued as an EPSG guidance note.
11	November 1998	Polynomial for Spain and Tunisia Mining Grid methods added.
12	February 1999	Abridged Molodensky formulas corrected.
13	July 1999	Lambert Conic Near Conformal and American Polyconic methods added.
14	December 1999	Stereographic and Tunisia Mining Grid formulas corrected. Krovak method added.
15	June 2000	General Polynomial and Affine methods added
16	December 2000	Lambert Conformal (Belgium) remarks revised; Oblique Mercator methods consolidated and formulas added. Similarity Transformation reversibility remarks amended.
17	June 2001	Lambert Conformal, Mercator and Helmert formulas corrected.
18	August 2002	Revised to include ISO 19111 terminology. Section numbering revised. Added Preface. Lambert Conformal (West Orientated), Lambert Azimuthal Equal Area, Albers, Equidistant Cylindrical (Plate Carrée), TM zoned, Bonne, Molodensky-Badedas methods added. Errors in Transverse Mercator (South Orientated) formula corrected.
19	December 2002	Polynomial formulas amended. Formula for spherical radius in Equidistant Cylindrical projection amended. Formula for Krovak projection amended. Degree representation conversions added. Editorial amendments made to subscripts and superscripts.
20	May 2003	Font for Greek symbols in Albers section amended.
21	October 2003	Typographic errors in example for Lambert Conic (Belgium) corrected. Polar Stereographic formulae extended for secant variants. General polynomial extended to degree 13. Added Abridged Molodensky and Lambert Azimuthal Equal Area examples and Reversible polynomial formulae.
22	December 2003	Errors in FE and FN values in example for Lambert Azimuthal Equal Area corrected.
23	January 2004	Database codes for Polar Stereographic variants corrected. Degree representation conversions withdrawn.
24	October 2004 From this revision, published as part 2 of a two-part set.	Corrected equation for u in Oblique Mercator. Added Guam projection, Geographic 3D to 2D conversion, vertical offset and gradient method, geoid models, bilinear interpolation methods. Added tables giving projection parameter definitions. Amended Molodensky-Badekas method name and added example. Added section on reversibility to Helmert 7-parameter transformations. Transformation section 2 reordered. Section 3 (concatenated operations) added.
25	May 2005	Amended reverse formulas for Lambert Conic Near-Conformal. Corrected Lambert Azimuthal Equal Area formulae. Symbol for latitude of pseudo standard parallel parameter made consistent. Corrected Affine Orthogonal Geometric transformation reverse example. Added Modified Azimuthal Equidistant projection.
26	July 2005	Further correction to Lambert Azimuthal Equal Area formulae. Correction to Moldenski-Badekas example.
27	September 2005	Miscellaneous linear coordinate operations paragraphs re-written to include reversibility and UKOOA P6. Improved formula for r' in Lambert Conic Near-Conformal.
28	November 2005	Corrected error in formula for t and false grid coordinates of 2SP example in Mercator projection.
29	April 2006	Typographic errors corrected. (For oblique stereographic, corrected formula for w. For Lambert azimuthal equal area, changed example. For Albers equal area, corrected formulae for alpha. For modified azimuthal equidistant, corrected formula for c. For Krovak, corrected formula for theta', clarified formulae for tO and lat. For Cassini, in example corrected radian value of longitude of natural origin). References to EPSG updated.

30	June 2006	Added Hyperbolic Cassini-Soldner. Corrected FE and FN values in example for Modified Azimuthal Equidistant. Added note to Krovak. Amended Abridged Molodensky description, corrected example.
31	August 2006	Corrected sign of value for G in Modified Azimuthal Equidistant example.
32	February 2007	Descriptive text for Oblique Mercator amended; formula for Laborde projection for Madagascar added. Added polar aspect equations for Lambert Azimuthal Equal Area. Corrected example in polynomial transformation for Spain. For Lambert 1SP, corrected equation for r'.
33	March 2007	For Krovak example, corrected axis names.
34	July 2007	Note on longitude wrap-around added prior to preample to map projection formulas, section 1.4. For Laborde, corrected formula for q'. For Albers Equal Area, corrected formulae for α and ß'.
35	April 2008	Longitude wrap-around note clarified. For Oblique Mercator, corrected symbol in formula for longitude. For Krovak, clarified defining parameters. Amended Vertical Offset description and formula. Added geographic/topocentric conversions, geocentric/topocentric conversions, Vertical Perspective, Orthographic, Lambert Cylindrical Equal Area, ellipsoidal development of Equidistant Cylindrical. Removed section on identification of map projection method.
36	July 2008	For Lambert Conic Near Conformal, corrected equations for φ'.
37	August 2008	Corrected general polynomial example.
38	January 2009	For Mercator (1SP), clarified use of φ_O. For Molodensky-Badekas, augmented example.Added Popular Visualisation Pseudo Mercator method, added formulas and examples for Mercator (Spherical) and formulas for American Polyconic.
39	April 2009	Preface revised to be consistent with other parts of GN7. For Lambert Azimuth Equal Area, in example corrected symbol for ß'. For Krovak, corrected formulas. For Equidistant Cylindrical (spherical) corrected fomula for R; comments on R added to all spherical methods. For Equidistant Cylindrical updated formula to hamonise parameters and symbols with similar methods.

1 Map projections and their coordinate conversion formulas

1.1 Introduction

Setting aside the large number of map projection methods which may be employed for atlas maps, equally small scale illustrative exploration maps, and wall maps of the world or continental areas, the EPSG dataset provides reference parameter values for orthomorphic or conformal map projections which are used for medium or large scale topographic or exploration mapping. Here accurate positions are important and sometimes users may wish to scale accurate positions, distances or areas from the maps.

Small scale maps normally assume a spherical earth and the inaccuracies inherent in this assumption are of no consequence at the usual scale of these maps. For medium and large scale sheet maps, or maps and coordinates held digitally to a high accuracy, it is essential that due regard is paid to the actual shape of the Earth. Such coordinate reference systems are therefore invariably based on an ellipsoid and its derived map projections. The EPSG dataset and this supporting conversion documentation considers only map projections for the ellipsoid.

Though not exhaustive the following list of named map projection methods are those which are most frequently encountered for medium and large scale mapping, some of them much less frequently than others since they are designed to serve only one particular country. They are grouped according to their possession of similar properties, which will be explained later. Except where indicated all are conformal.

Mercator	Cylindrical
with one standard parallel	
with two standard parallels	
Cassini-Soldner (N.B. not conformal)	Transverse Cylindrical
Transverse Mercator Group	Transverse Cylindrical
Transverse Mercator (including south oriented version)	
Universal Transverse Mercator	
Gauss-Kruger	
Gauss-Boaga	
Oblique Mercator Group	Oblique Cylindrical
Hotine Oblique Mercator	
Oblique Mercator	
Laborde Oblique Mercator	
Lambert Conical Conformal	Conical
with one standard parallel	
with two standard parallels	
Stereographic	Azimuthal
Polar	
Oblique and equatorial	

1.2 Map Projection parameters

A map projection grid is related to the geographical graticule of an ellipsoid through the definition of a coordinate conversion method and a set of parameters appropriate to that method. Different conversion methods may require different parameters. Any one coordinate conversion method may take several different sets of associated parameter values, each set related to a particular map projection zone applying to a particular country or area of the world. Before setting out the formulas involving these parameters, which enable the coordinate conversions for the projection methods listed above, it is as well to understand the nature of the parameters.

The plane of the map and the ellipsoid surface may be assumed to have one particular point in common. This point is referred to as the **natural origin**. It is the point from which the values of both the geographical coordinates on the ellipsoid and the grid coordinates on the projection are deemed to increment or decrement for computational purposes. Alternatively it may be considered as the point which in the absence of application of false coordinates has grid coordinates of (0,0). For example, for projected coordinate reference systems using the Cassini-Soldner or Transverse Mercator methods, the natural origin is at the intersection of a chosen parallel and a chosen meridian (see Figure 2 at end of section). The chosen parallel will frequently but not necessarily be the equator. The chosen meridian will usually be central to the mapped area.. For the stereographic projection the origin is at the centre of the projection where the plane of the map is imagined to be tangential to the ellipsoid.

Since the natural origin may be at or near the centre of the projection and under normal coordinate circumstances would thus give rise to negative coordinates over parts of the map, this origin is usually given false coordinates which are large enough to avoid this inconvenience. Hence each natural origin will normally have **False Easting, FE** and **False Northing, FN** values**.** For example, the false easting for the origins of all Universal Transverse Mercator zones is 500000m. As the UTM origin lies on the equator, areas north of the equator do not need and are not given a false northing but for mapping southern hemisphere areas the equator origin is given a false northing of 10,000,000m, thus ensuring that no point in the southern hemisphere will take a negative northing coordinate. Figure 4 illustrates the UTM arrangements.

These arrangements suggest that if there are false easting and false northing for the real or natural origin, there is also a **Grid Origin** which has coordinates (0,0). In general this point is of no consequence though its geographic position may be computed if needed. For example, for the WGS 84 / UTM zone 31N coordinate reference system which has a natural origin at 0°N, 3°E where false easting is 500000m E (and false northing is 0m N), the grid origin is at 0°N, 1°29'19.478"W. Sometimes however, rather than base the easting and northing coordinate reference system on the natural origin by giving it **FE** and **FN** values, it may be convenient to select a **False Origin** at a specific meridian/parallel intersection and attribute the false coordinates **Easting at False Origin, E_F** and **Northing at False Origin, N_F** to this. The related easting and northing of the natural origin may then be computed if required.

The natural origin will always lie on a meridian of longitude. Longitudes are most commonly expressed relative to the **Prime Meridian** of Greenwich but some countries, particularly in former times, have preferred to relate their longitudes to a prime meridian through their national astronomic observatory, usually sited in or near their capital city, e.g. Paris for France, Bogota for Colombia. The meridian of the projection zone origin is known as the **Longitude of Origin**. For certain projection types it is often termed the **Central Meridian** or abbreviated as **CM** and provides the direction of the northing axis of the projected coordinate reference system**.**

Because of the steadily increasing distortion in the scale of the map with increasing distance from the origin, central meridian or other line on which the scale is the nominal scale of the projection, it is usual to limit the extent of a projection to within a few degrees of latitude or longitude of this point or line. Thus, for example, a UTM or other Transverse Mercator projection zone will normally extend only 2 or 3 degrees from the central meridian. Beyond this area another **zone** of the projection, with a new origin and central meridian, needs to be used or created. The UTM system has a specified 60 numbered zones, each 6 degrees wide, covering the ellipsoid between the 84 degree North and 80 degree South latitude parallels. Other Transverse Mercator projection zones may be constructed with different central meridians, and different origins chosen to suit the countries or states for which they are used. A number of these are included in the EPSG dataset. Similarly a Lambert Conic Conformal zone distorts most rapidly in the north-south direction and may, as in Texas, be divided into latitudinal bands.

In order to further limit the scale distortion within the coverage of the zone or projection area, some projections introduce a **scale factor** at the origin (on the central meridian for Transverse Mercator projections), which has the effect of reducing the nominal scale of the map here and making it have the nominal scale some distance away. For example in the case of the UTM and some other Transverse Mercator projections a scale factor of slightly less than unity is introduced on the central meridian thus making it unity on two meridians either side of the central one, and reducing its departure from unity beyond these. The scale factor is a required parameter whether or not it is unity and is usually symbolised as $\mathbf{k_O}$.

Thus for projections in the Transverse Mercator group in section 1.1 above, the parameters which are required to completely and unambiguously define the projection method are:

Latitude of natural origin
Longitude of natural origin (the central meridian)
Scale factor at natural origin (on the central meridian)
False easting
False northing

Since the UTM zones obey set rules, it is sufficient to state only the UTM zone number (or central meridian). The remaining parameters from the above list are defined by the rules.

It has been noted that the Transverse Mercator projection is employed for the topographical mapping of longitudinal bands of territories, limiting the amount of scale distortion by limiting the extent of the projection either side of the central meridian. Sometimes the shape, general trend and extent of some countries makes it preferable to apply a single zone of the same kind of projection but with its central line aligned with the trend of the territory concerned rather than with a meridian. So, instead of a meridian forming this true scale central line for one of the various forms of Transverse Mercator, or the equator forming the line for the Mercator, a line with a particular azimuth traversing the territory is chosen, and the same principles of construction are applied to derive what is now an Oblique Mercator. This projection is sometimes referred to as the Hotine Oblique Mercator after the British geodesist who set out its formulas for application to Malaysian Borneo (East Malaysia) and also West Malaysia. Laborde had previously developed the projection system for Madagascar, and Switzerland uses a similar system derived by Rosenmund.

More recently (1974) Lee has derived formulas for a minimum scale factor projection for New Zealand known as the New Zealand Map Grid. The line of minimum scale follows the general alignment of the two main islands. This resembles an Oblique Mercator projection in its effect, but is not strictly an Oblique Mercator. The additional mathematical complexity of the projection enables its derivation via an Oblique Stereographic projection, which is sometimes the way it is classified. Because of its unique formulation inclusion of the New Zealand Map Grid within international mapping software was sporadic; as a consequence New Zealand has reverted to the frequently-encountered Transverse Mercator for its most recent mapping.

The parameters required to define an Oblique Mercator projection are:

Latitude of projection centre (the origin point on the initial line)
Longitude of projection centre
Azimuth of initial line [at the projection centre]
Scale factor on initial line
Angle from Rectified to Skewed grid

and then either

False easting (easting at the projection natural origin)
False northing (northing at the projection natural origin)

or

Easting at projection centre
Northing at projection centre

It is possible to define the azimuth of the initial line through the latitude and longitude of two widely spaced points along that line. This approach is not followed in the EPSG dataset.

For **Conical** map projections, which for the normal aspect may be considered as the projection of the ellipsoid onto an enveloping cone in contact with the ellipsoid along a parallel of latitude, the parallel of contact is known as **a standard parallel** and the scale is regarded as true along this parallel. Sometimes the cone is imagined to cut the ellipsoid with coincidence of the two surfaces along **two standard parallels**. All other parallels will be concentric with the chosen standard parallel or parallels but for the Lambert Conical Conformal will have varying separations to preserve the conformal property. All meridians will radiate with equal angular separations from the centre of the parallel circles but will be compressed from the 360 longitude degrees of the ellipsoid to a sector whose angular extent depends on the chosen standard parallel, - or both standard parallels if there are two. Of course the normal longitudinal extent of the projection will depend on the extent of the territory to be projected and will never approach 360 degrees.

As in the case of the Transverse Mercator above it is sometimes desirable to limit the maximum positive scale distortion for the one standard parallel case by distributing it more evenly over the extent of the mapped area. This may be achieved by introducing a scale factor on the standard parallel of slightly less than unity thus making it unity on two parallels either side of it. This achieves the same effect as choosing two specific standard parallels in the first place, on which the nominal scale will be preserved. The projection is then a Lambert Conical Conformal projection with two standard parallels**.** Although, strictly speaking, the scale on a standard parallel is always the nominal scale of the map and the scale factor on the one or two standard parallels should be unity, it is sometimes convenient to consider a Lambert Conical Conformal projection with one standard parallel yet which has a scale factor on the standard parallel of less than unity. This provision is allowed for in the EPSG dataset, where the single standard parallel is referred to as the **latitude of natural origin**. For an ellipsoidal projection the natural origin will fall slightly poleward of the mean of the latitudes of the two standard parallels.

A **longitude of origin** or **central meridian** will again be chosen to bisect the area of the map or, more usually, the total national map area for the country or state concerned. Where this cuts the one standard parallel will be the natural origin of the projected coordinate reference system and, as for the Transverse Mercator, it will be given a **False easting and False northing** to ensure that there are no negative coordinates within the projected area (see Figure 5). Where two standard parallels are specified a false origin may be chosen at the intersection of a specific parallel with the central meridian. This point will be given an **easting at false origin** and a **northing at false origin** to ensure that no negative coordinates will result. Figure 6 illustrates these arrangements.

It is clear that any number of Lambert projection zones may be formed according to which standard parallel or standard parallels are chosen and this is clearly exemplified by those which are used for many of the United States State Plane coordinate zones. They are normally chosen either, for one standard parallel, to approximately bisect the latitudinal extent of the country or area or, for two standard parallels, to embrace most of the latitudinal extent of the area. In the latter case the aim is to minimise the maximum scale distortion which will affect the mapped area and various formulas have been developed by different mathematicians to select the appropriate standard parallels to achieve this. Kavraisky was one mathematician who derived a recipe for choosing the standard parallels to achieve minimal scale distortion. But however the selection of the standard parallels is made the same projection formulas apply. Thus the parameters needed to specify a projection in the Lambert projection will be as follows:

For a Lambert Conical Conformal with one standard parallel (1SP),
- Latitude of natural origin (the Standard Parallel)
- Longitude of natural origin (the Central Meridian)
- Scale factor at natural origin (on the Standard Parallel)
- False easting
- False northing

For a Lambert Conical Conformal with two standard parallels (2SP),
- Latitude of false origin
- Longitude of false origin (the Central Meridian)
- Latitude of first standard parallel
- Latitude of second standard parallel
- Easting at false origin
- Northing at false origin

where the order of the standard parallels is not material if using the formulas which follow.

The limiting case of the Lambert Conic Conformal having the apex of the cone at infinity produces a **cylindrical** projection, the Mercator. Here, for the single standard parallel case the latitude of natural origin is the equator. For the two standard parallel case the two parallels have equal latitude in the north and south hemispheres. In both one and two standard parallel cases, grid coordinates are for the natural origin at the intersection of the equator and the central meridian (see figure 1). Thus the parameters needed to specify a map projection using the Mercator map projection method will be:

For a Mercator with one standard parallel (1SP),
- Latitude of natural origin (always the Equator, documented only for completeness[1])
- Longitude of natural origin (the Central Meridian)
- Scale factor at natural origin (on the Equator)
- False easting
- False northing

For a Mercator with two standard parallels (2SP),
- Latitude of first standard parallel[2]
- Longitude of natural origin (the Central Meridian)
- False easting (grid coordinate at the intersection of the CM with the equator)
- False northing

For **Azimuthal** map projections**,** which are only infrequently used for ellipsoidal topographic mapping purposes, the natural origin will be at the centre of the projection where the map plane is imagined to be tangential to the ellipsoid and which will lie at the centre of the area to be projected. The central meridian will pass through the natural origin. This point will be given a False Easting and False Northing.

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[1] In the formulas that follow, the latitude of natural origin is not used. However for completeness in CRS labeling the EPSG dataset includes this parameter, which must have a value of zero.

[2] In the formulas that follow the absolute value of the first standard parallel must be used.

The parameters needed to specify the Stereographic map projection method are:

Latitude of natural origin
Longitude of natural origin (the central meridian for the oblique case)
Scale factor at natural origin
False easting
False northing

TABLE 1.
Parameters used in map projection conversions

Parameter Name	**Symbol**	**Description**
Angle from Rectified to Skew Grid	γ_C	The angle at the natural origin of an oblique projection through which the natural coordinate reference system is rotated to make the projection north axis parallel with true north.
Azimuth of initial line	α_C	The azimuthal direction (north zero, east of north being positive) of the great circle which is the centre line of an oblique projection. The azimuth is given at the projection center.
Central meridian		See Longitude of natural origin
Easting at false origin	E_F	The easting value assigned to the false origin.
Easting at projection centre	E_C	The easting value assigned to the projection centre.
False easting	FE	The value assigned to the abscissa (east or west) axis of the projection grid at the natural origin.
False northing	FN	The value assigned to the ordinate (north or south) axis of the projection grid at the natural origin.
False origin		A specific parallel/meridian intersection other than the natural origin to which the grid coordinates E_F and N_F, are assigned.
Grid origin		The point which has coordinates (0,0). It is offset from the natural origin by the false easting and false northing. In some projection methods it may alternatively be offset from the false origin by Easting at false origin and Northing at false origin. In general this point is of no consequence.
Initial line		The line on the surface of the earth model which forms the axis for the grid of an oblique projection.
Initial longitude	λ_I	The longitude of the western limit of the first zone of a Transverse Mercator zoned grid system.
Latitude of 1st standard parallel	φ_1	For a conic projection with two standard parallels, this is the latitude of one of the parallels at which the cone intersects with the ellipsoid. It is normally but not necessarily that nearest to the pole. Scale is true along this parallel.
Latitude of 2nd standard parallel	φ_2	For a conic projection with two standard parallels, this is the latitude of one of the parallels at which the cone intersects with the ellipsoid. It is normally but not necessarily that nearest to the equator. Scale is true along this parallel.
Latitude of false origin	φ_F	The latitude of the point which is not the natural origin and at which grid coordinate values easting at false origin and northing at false origin are defined.

Parameter Name	**Symbol**	**Description**
Latitude of natural origin	φ_O	The latitude of the point from which the values of both the geographical coordinates on the ellipsoid and the grid coordinates on the projection are deemed to increment or decrement for computational purposes. Alternatively it may be considered as the latitude of the point which in the absence of application of false coordinates has grid coordinates of (0,0).
Latitude of projection centre	φ_C	For an oblique projection, this is the latitude of the point at which the azimuth of the initial line is defined.
Latitude of pseudo standard parallel	φ_P	Latitude of the parallel on which the conic or cylindrical projection is based. This latitude is not geographic, but is defined on the conformal sphere AFTER its rotation to obtain the oblique aspect of the projection.
Latitude of standard parallel	φ_F	For polar aspect azimuthal projections, the parallel on which the scale factor is defined to be unity.
Longitude of false origin	λ_F	The longitude of the point which is not the natural origin and at which grid coordinate values easting at false origin and northing at false origin are defined.
Longitude of natural origin	λ_O	The longitude of the point from which the values of both the geographical coordinates on the ellipsoid and the grid coordinates on the projection are deemed to increment or decrement for computational purposes. Alternatively it may be considered as the longitude of the point which in the absence of application of false coordinates has grid coordinates of (0,0). Sometimes known as "central meridian (CM)".
Longitude of origin	λ_O	For polar aspect azimuthal projections, the meridian along which the northing axis increments and also across which parallels of latitude increment towards the north pole.
Longitude of projection centre	λ_C	For an oblique projection, this is the longitude of the point at which the azimuth of the initial line is defined.
Natural origin		The point from which the values of both the geographical coordinates on the ellipsoid and the grid coordinates on the projection are deemed to increment or decrement for computational purposes. Alternatively it may be considered as the point which in the absence of application of false coordinates has grid coordinates of (0,0). For example, for projected coordinate reference systems using the Transverse Mercator method, the natural origin is at the intersection of a chosen parallel and a chosen central meridian.
Northing at false origin	N_F	The northing value assigned to the false origin.
Northing at projection centre	N_C	The northing value assigned to the projection centre.
Origin		See natural origin, false origin and grid origin.
Projection centre		On an oblique cylindrical or conical projection, the point at which the direction of the cylinder or cone and false coordinates are defined.
Scale factor at natural origin	k_O	The factor by which the map grid is reduced or enlarged during the projection process, defined by its value at the natural origin.
Scale factor on initial line	k_C	The factor by which an oblique projection's map grid is reduced or enlarged during the projection process, defined by its value along the centre line of the cylinder or cone.
Scale factor on pseudo standard parallel	k_P	The factor by which the map grid is reduced or enlarged during the projection process, defined by its value at the pseudo-standard parallel.

Parameter Name	**Symbol**	**Description**
Zone width	W	The longitude width of a zone of a Transverse Mercator zoned grid system.

TABLE 2

Summary of Coordinate Operation Parameters required for some Map Projections

Coordinate Operation Parameter Name	**Coordinate Operation Method**								
	Mercator (1SP)	Mercator (2SP)	Cassini-Soldner	Transverse Mercator	Hotine Oblique Mercator	Oblique Mercator	Lambert Conical (1 SP)	Lambert Conical (2 SP)	Oblique Stereo-graphic
Latitude of false origin								1	
Longitude of false origin								2	
Latitude of 1st standard parallel		1						3	
Latitude of 2nd standard parallel								4	
Easting at false origin								5	
Northing at false origin								6	
Latitude of projection centre					1	1			
Longitude of projection centre					2	2			
Scale factor on initial line					3	3			
Azimuth of initial line					4	4			
Angle from Rectified to Skewed grid					5	5			
Easting at projection centre						6			
Northing at projection centre						7			
Latitude of natural origin	1 =equator		1	1			1		1
Longitude of natural origin	2	2	2	2			2		2
Scale factor at natural origin	3			3			3		3
False easting	4	3	3	4	6		4		4
False northing	5	4	4	5	7		5		5

TABLE 3

Ellipsoid parameters used in conversions and transformations

In the formulas in this Guidance Note the basic ellipsoidal parameters are represented by symbols and derived as follows:

Primary ellipsoid parameters		
Parameter Name	Symbol	Description
semi-major axis	a	Length of the semi-major axis of the ellipsoid, the radius of the equator.
semi-minor axis	b	Length of the semi-minor axis of the ellipsoid, the distance along the ellipsoid axis between equator and pole.
inverse flattening	$1/f$	$= a/(a - b)$
Derived ellipsoid parameters		
Parameter Name	Symbol	Description
flattening	f	$= 1 / (1/f)$
eccentricity	e	$= \sqrt{(2f - f^2)}$
second eccentricity	e'	$= \sqrt{[e^2 / (1 - e^2)]}$
radius of curvature in the meridian	ρ	radius of curvature of the ellipsoid in the plane of the meridian at latitude φ, where $\rho = a(1 - e^2)/(1 - e^2\sin^2\varphi)^{3/2}$
radius of curvature in the prime vertical	ν	radius of curvature of the ellipsoid perpendicular to the meridian at latitude φ, where $\nu = a / (1 - e^2\sin^2\varphi)^{1/2}$
radius of authalic sphere	R_A	radius of sphere having same surface area as ellipsoid. $R_A = a * [(1 - \{(1 - e^2) / (2 - e)\} * \{LN[(1 - e) / (1 + e)]\}) * 0.5]^{0.5}$
radius of conformal sphere	R_C	$= \sqrt{(\rho\ \nu)} = [a \sqrt{(1 - e^2)} / (1 - e^2 \sin^2\varphi)$ This is a function of latitude and therefore not constant. When used for spherical projections the use of φ_O (or φ_1 as relevant to method) for φ is suggested, except if the projection is equal area when R_A (see above) should be used.

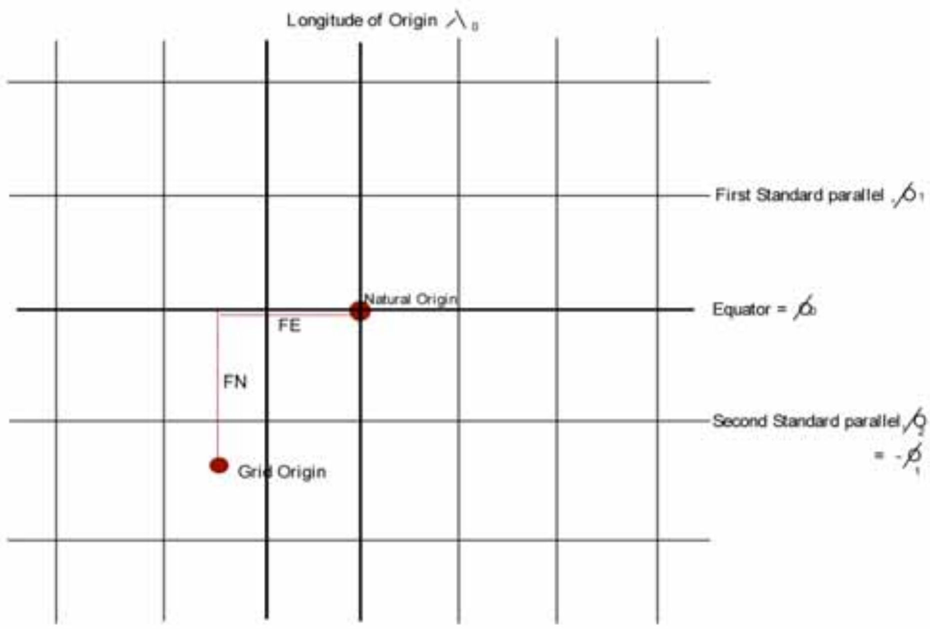

Figure 1. Key Diagram for Mercator Projection arrangements
(One and two standard parallel cases)

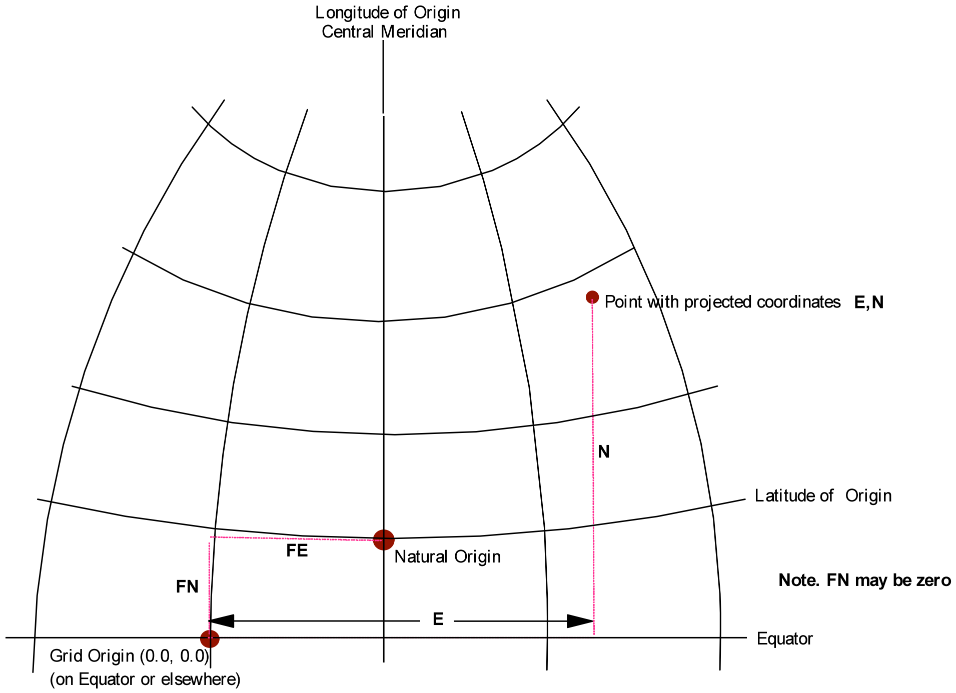

Figure 2. Key Diagram for Transverse Mercator Projection arrangements (N.Hemisphere)

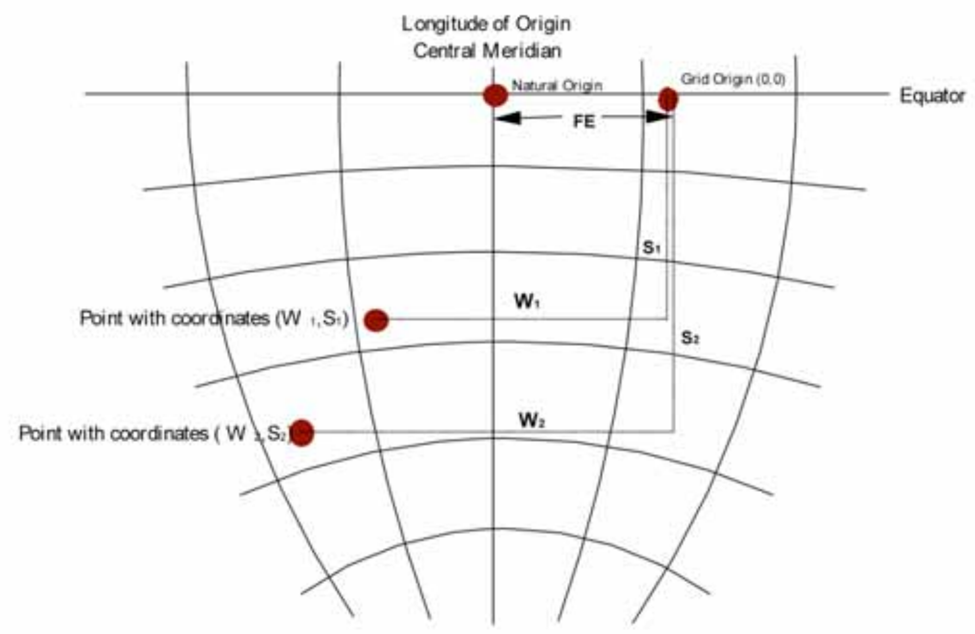

Figure 3. Key Diagram for South oriented Transverse Mercator Projection arrangements

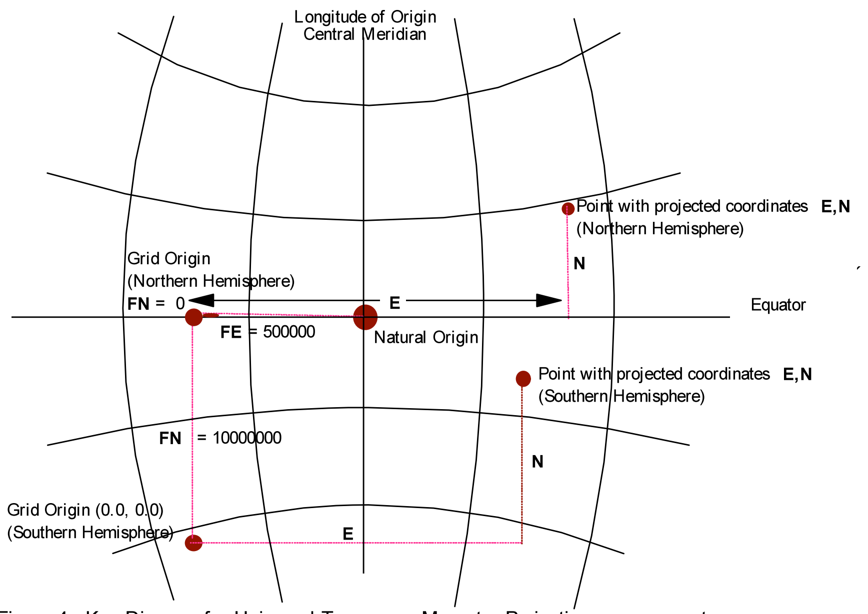

Figure 4. Key Diagram for Universal Transverse Mercator Projection arrangements
(N and S hemisphere cases)

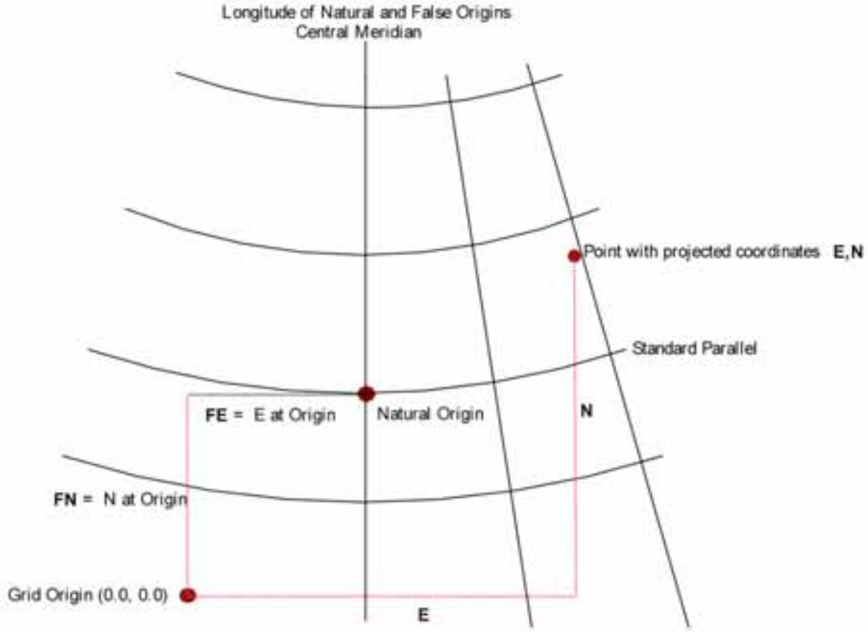

Figure 5. Key Diagram for Lambert Conical Conformal Projection with one standard parallel

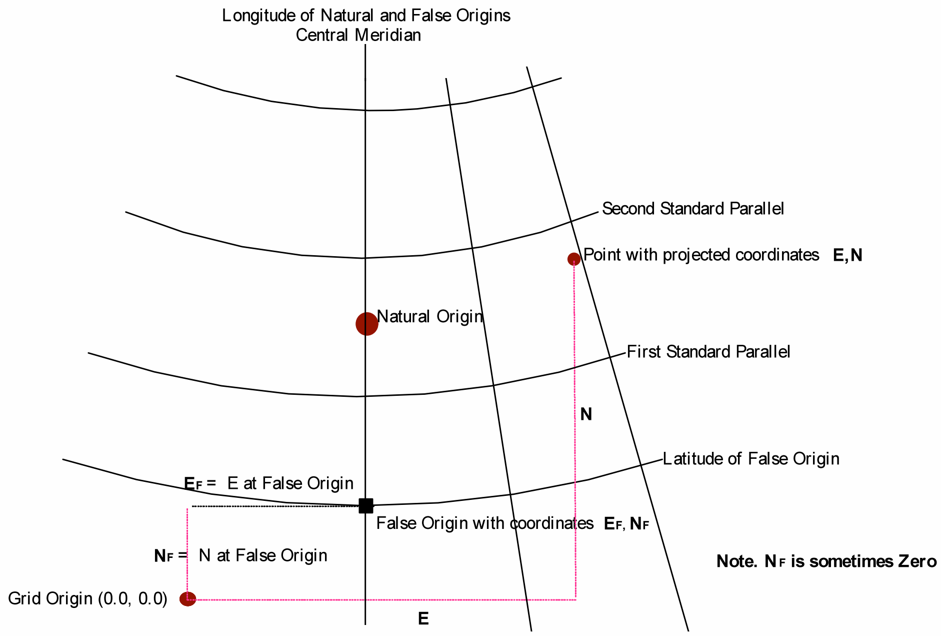

Figure 6. Key Diagram for Lambert Conical Conformal Projection with two standard parallels

1.3 Map Projection formulas

In general, only formulas for computation on the ellipsoid are considered. Projection formulas for the spherical earth are simpler but the spherical figure is inadequate to represent positional data with great accuracy at large map scales for the real earth. Projections of the sphere are only suitable for illustrative maps at scale of 1:1 million or less where precise positional definition is not critical.

The formulas which follow are largely adapted from "Map Projections - A Working Manual" by J.P.Snyder, published by the U.S. Geological Survey as Professional Paper No.1395[3]. As well as providing an extensive overview of most map projections in current general use, and the formulas for their construction for both the spherical and ellipsoidal earth, this excellent publication provides computational hints and details of the accuracies attainable by the formulas. It is strongly recommended that all those who have to deal with map projections for medium and large scale mapping should follow its guidance.

There are a number of different formulas available in the literature for map projections other than those quoted by Snyder. Some are closed formulas; others, for ease of calculation, may depend on series expansions and their precision will generally depend on the number of terms used for computation. Generally those formulas which follow in this chapter will provide results which are accurate to within a decimetre, which is normally adequate for exploration mapping purposes. Coordinate expression and computations for engineering operations are usually consistently performed in grid terms.

The importance of one further variable should be noted. This is the unit of linear measurement used in the definition of projected coordinate reference systems. For metric map projections the unit of measurement is restricted to this unit. For non-metric map projections the metric ellipsoid semi-major axis needs to be converted to the projected coordinate reference system linear unit before use in the formulas below. The relevant ellipsoid is obtained through the datum part of the projected coordinate reference system.

Reversibility
Different formulas are required for forward and reverse map projection conversions: the forward formula cannot be used for the reverse conversion. However both forward and reverse formulas are explicitly given in the sections below as parts of a single conversion method. As such, map projection methods are described in the EPSG dataset as being reversible. Forward and reverse formulas for each conversion method use the projection parameters appropriate to that method with parameter values unchanged.

Longitude 'wrap-around'
The formulas that follow assume longitudes are described using the range $-180 \leq \lambda \leq +180$ degrees. If the area of interest crosses the 180° meridian and an alternative longitude range convention is being used, longitudes need to be converted to fall into this $-180 \leq \lambda \leq +180$ degrees range. This may be achieved by applying the following:

If $(\lambda - \lambda_O) \leq -180°$ then $\lambda = \lambda + 360°$. This may be required when $\lambda_O > 0°$.
If $(\lambda - \lambda_O) \geq 180°$ then $\lambda = \lambda - 360°$. This may be required when $\lambda_O < 0°$.

In the formulas that follow the symbol λ_C or λ_F may be used rather than λ_O, but the same principle applies.

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[3] This was originally published with the title "Map Projections Used by the US Geological Survey". In some cases the formulas given are insufficient for global use. In these cases EPSG has modified the formulas. Note that the origin of most map projections is given false coordinates (FE and FN or E_F and N_F or E_C and N_C) to avoid negative coordinates. In the EPSG formulas these values are included where appropriate so that the projected coordinates of points result directly from the quoted formulas.

1.3.1 Lambert Conic Conformal

For territories with limited latitudinal extent but wide longitudinal width it may sometimes be preferred to use a single projection rather than several bands or zones of a Transverse Mercator. The Lambert Conic Conformal may often be adopted in these circumstances. But if the latitudinal extent is also large there may still be a need to use two or more zones if the scale distortion at the extremities of the one zone becomes too large to be tolerable.

Conical projections with one standard parallel are normally considered to maintain the nominal map scale along the parallel of latitude which is the line of contact between the imagined cone and the ellipsoid. For a one standard parallel Lambert the natural origin of the projected coordinate system is the intersection of the standard parallel with the longitude of origin (central meridian). See Figure 5 at end of section 1.3. To maintain the conformal property the spacing of the parallels is variable and increases with increasing distance from the standard parallel, while the meridians are all straight lines radiating from a point on the prolongation of the ellipsoid's minor axis.

Sometimes however, although a one standard parallel Lambert is normally considered to have unity scale factor on the standard parallel, a scale factor of slightly less than unity is introduced on this parallel. This is a regular feature of the mapping of some former French territories and has the effect of making the scale factor unity on two other parallels either side of the standard parallel. The projection thus, strictly speaking, becomes a Lambert Conic Conformal projection with **two** standard parallels. From the one standard parallel and its scale factor it is possible to derive the equivalent two standard parallels and then treat the projection as a two standard parallel Lambert conical conformal, but this procedure is seldom adopted. Since the two parallels obtained in this way will generally not have integer values of degrees or degrees minutes and seconds it is instead usually preferred to select two specific parallels on which the scale factor is to be unity, as for several State Plane Coordinate systems in the United States.

The choice of the two standard parallels will usually be made according to the latitudinal extent of the area which it is wished to map, the parallels usually being chosen so that they each lie a proportion inboard of the north and south margins of the mapped area. Various schemes and formulas have been developed to make this selection such that the maximum scale distortion within the mapped area is minimised, e.g. Kavraisky in 1934, but whatever two standard parallels are adopted the formulas are the same.

1.3.1.1 Lambert Conic Conformal (2SP)

(EPSG dataset coordinate operation method code 9802)

To derive the projected Easting and Northing coordinates of a point with geographical coordinates (φ,λ) the formulas for the Lambert Conic Conformal **two standard parallel case** (EPSG datset coordinate operation method code 9802) are:

Easting, $E = E_F + r \sin\theta$
Northing, $N = N_F + r_F - r \cos\theta$

where $m = \cos\varphi/(1 - e^2\sin^2\varphi)^{0.5}$ for m_1, φ_1, and m_2, φ_2 where φ_1 and φ_2 are the latitudes of the standard parallels

$t = \tan(\pi/4 - \varphi/2)/[(1 - e \sin\varphi)/(1 + e \sin\varphi)]^{e/2}$ for t_1, t_2, t_F and t using φ_1, φ_2, φ_F and φ respectively

$n = (\ln m_1 - \ln m_2)/(\ln t_1 - \ln t_2)$

$F = m_1/(n t_1^n)$

$r = a F t^n$ for r_F and r, where r_F is the radius of the parallel of latitude of the false origin

$\theta = n(\lambda - \lambda_F)$

The reverse formulas to derive the latitude and longitude of a point from its Easting and Northing values are:

$\varphi = \pi/2 - 2\mathrm{atan}\{t'[(1 - e\sin\varphi)/(1 + e\sin\varphi)]^{e/2}\}$
$\lambda = \theta'/n + \lambda_F$

where

$r' = \pm\{(E - E_F)^2 + [r_F - (N - N_F)]^2\}^{0.5}$, taking the sign of n
$t' = (r'/(aF))^{1/n}$
$\theta' = \mathrm{atan}\,[(E - E_F)/(r_F - (N - N_F))]$

and n, F, and r_F are derived as for the forward calculation.

Note that the formula for φ requires iteration. First calculate t' and then a trial value for φ using $\varphi = \pi/2 - 2\mathrm{atan}\ t'$. Then use the full equation for φ substituting the trial value into the right hand side of the equation. Thus derive a new value for φ. Iterate the process until φ does not change significantly. The solution should quickly converge, in 3 or 4 iterations.

Example:
For Projected Coordinate Reference System: NAD27 / Texas South Central

Parameters:

Ellipsoid:	Clarke 1866	a = 6378206.400 metres	=	20925832.16 US survey feet
		1/f = 294.97870		
	then	e = 0.08227185		e^2 = 0.00676866

Latitude of false origin	φ_F	27°50'00"N	=	0.48578331 rad
Longitude of false origin	λ_F	99°00'00"W	=	-1.72787596 rad
Latitude of 1st standard parallel	φ_1	28°23'00"N	=	0.49538262 rad
Latitude of 2nd standard parallel	φ_2	30°17'00"N	=	0.52854388 rad
Easting at false origin	E_F	2000000.00	US survey feet	
Northing at false origin	N_F	0.00	US survey feet	

Forward calculation for:

Latitude	φ	=	28°30'00.00"N	=	0.49741884 rad
Longitude	λ	=	96°00'00.00"W	=	-1.67551608 rad

first gives :

m_1	=	0.88046050	m_2	=	0.86428642
t	=	0.59686306	t_F	=	0.60475101
t_1	=	0.59823957	t_2	=	0.57602212
n	=	0.48991263	F	=	2.31154807
r	=	37565039.86	r_F	=	37807441.20
θ	=	0.02565177			

Then	Easting	E	=	2963503.91 US survey feet
	Northing	N	=	254759.80 US survey feet

Reverse calculation for same easting and northing first gives:

θ'	=	0.025651765
t'	=	0.59686306
r'	=	37565039.86

Then	Latitude	φ	=	28°30'00.000"N
	Longitude	λ	=	96°00'00.000"W

1.3.1.2 Lambert Conic Conformal (1SP)

(EPSG dataset coordinate operation method code 9801)

The formulas for the two standard parallel can be used for the Lambert Conic Conformal single standard parallel case (EPSG dataset coordinate operation method code 9801) with minor modifications. Then

$E = FE + r \sin\theta$

$N = FN + r_O - r \cos\theta$, using the natural origin rather than the false origin.

where

$n = \sin \varphi_O$

$r = a F t^n k_O$ for r_O, and r

t is found for t_O, φ_O and t, φ and m, F, and θ are found as for the two standard parallel case.

The reverse formulas for φ and λ are as for the two standard parallel case above, with n, F and r_O as before and

$\theta' = \operatorname{atan}\{(E - FE)/[r_O - (N - FN)]\}$

$r' = \pm\{(E - FE)^2 + [r_O - (N - FN)]^2\}^{0.5}$, taking the sign of n

$t' = (r'/(a\, k_O\, F))^{1/n}$

Example:

For Projected Coordinate Reference System: JAD69 / Jamaica National Grid

Parameters:

Ellipsoid:	Clarke 1866	a = 6378206.400 metres	1/f = 294.97870
	then	e = 0.08227185	e^2 = 0.00676866

Latitude of natural origin	φ_O	18°00'00"N	=	0.31415927 rad
Longitude of natural origin	λ_O	77°00'00"W	=	-1.34390352 rad
Scale factor at natural origin	k_O	1.000000		
False easting	FE	250000.00	metres	
False northing	FN	150000.00	metres	

Forward calculation for:

Latitude	φ	=	17°55'55.80"N	=	0.31297535 rad
Longitude	λ	=	76°56'37.26"W	=	-1.34292061 rad

first gives

m_O	=	0.95136402	t_O	=	0.72806411
F	=	3.39591092	n	=	0.30901699
R	=	19643955.26	r_O	=	19636447.86
θ	=	0.00030374	t	=	0.728965259

Then	Easting	E	=	255966.58 metres
	Northing	N	=	142493.51 metres

Reverse calculation for the same easting and northing first gives

θ'	=	0.000303736
t'	=	0.728965259
m_O	=	0.95136402
r'	=	19643955.26

Then	Latitude	φ	=	17°55'55.80"N
	Longitude	λ	=	76°56'37.26"W

1.3.1.3 Lambert Conic Conformal (West Orientated)

(EPSG dataset coordinate operation method code 9826)

In older mapping of Denmark and Greenland the Lambert Conic Conformal is used with axes positive north and **west**. To derive the projected Westing and Northing coordinates of a point with geographical coordinates (φ, λ) the formulas are as for the standard Lambert Conic Conformal (1SP) case above (EPSG dataset coordinate operation method code 9801) except for:

$$W = FE - r * \sin\theta$$

In this formula the term FE retains its definition, i.e. in the Lambert Conic Conformal (West Orientated) method it increases the Westing value at the natural origin. In this method it is effectively false westing (FW).

The reverse formulas to derive the latitude and longitude of a point from its Westing and Northing values are as for the standard Lambert Conic Conformal (1SP) case except for:

$\theta' = \text{atan}[(FE - W)/\{r_O - (N - FN)\}]$
$r' = +/-[(FE - W)^2 + \{r_O - (N - FN)\}^2]^{0.5}$, taking the sign of n

1.3.1.4 Lambert Conic Conformal (2 SP Belgium)

(EPSG dataset coordinate operation method code 9803)

In 1972, in order to retain approximately the same grid coordinates after a change of geodetic datum, a modified form of the two standard parallel case was introduced in Belgium. In 2000 this modification was replaced through use of the regular Lambert Conic Conformal (2 SP) map projection with appropriately modified parameter values.

In the 1972 modification the formulas for the regular Lambert Conic Conformal (2SP) case given above are used except for:

Easting, $E = E_F + r \sin(\theta - a)$
Northing, $N = N_F + r_F - r \cos(\theta - a)$

and for the reverse formulas

$\lambda = [(\theta' + a)/n] + \lambda_F$

where a = 29.2985 seconds.

Example:
For Projected Coordinate Reference System: Belge 1972 / Belge Lambert 72

Parameters:

Ellipsoid:	International 1924	a = 6378388 metres	1/f = 297.0
	then	e = 0.08199189	e^2 = 0.006722670

Latitude of false origin	φ_F	90°00'00"N	=	1.57079633 rad
Longitude of false origin	λ_F	4°21'24.983"E	=	0.07604294 rad
Latitude of 1st standard parallel	φ_1	49°50'00"N	=	0.86975574 rad
Latitude of 2nd standard parallel	φ_2	51°10'00"N	=	0.89302680 rad
Easting at false origin	E_F	150000.01	metres	
Northing at false origin	N_F	5400088.44	metres	

Forward calculation for:

Latitude	φ	=	50°40'46.461"N	=	0.88452540 rad
Longitude	λ	=	5°48'26.533"E	=	0.10135773 rad

first gives :

m_1	=	0.64628304	m_2	=	0.62834001
t	=	0.35913403	t_F	=	0.00
t_1	=	0.36750382	t_2	=	0.35433583
n	=	0.77164219	F	=	1.81329763
r	=	5248041.03	r_F	=	0.00
θ	=	0.01953396	a	=	0.00014204

Then	Easting	E	=	251763.20 metres
	Northing	N	=	153034.13 metres

Reverse calculation for same easting and northing first gives:

θ'	=	0.01939192
t'	=	0.35913403
r'	=	5248041.03

Then	Latitude	φ	=	50°40'46.461"N
	Longitude	λ	=	5°48'26.533"E

1.3.1.5 Lambert Conic Near-Conformal

(EPSG dataset coordinate operation method code 9817)

The Lambert Conformal Conic with one standard parallel formulas, as published by the Army Map Service, are still in use in several countries. The AMS uses series expansion formulas for ease of computation, as was normal before the electronic computer made such approximate methods unnecessary. Where the expansion series have been carried to enough terms the results are the same to the centimetre level as through the Lambert Conic Conformal (1SP) formulas above. However in some countries the expansion formulas were truncated to the third order and the map projection is not fully conformal. The full formulas are used in Libya but from 1915 for France, Morocco, Algeria, Tunisia and Syria the truncated formulas were used. In 1943 in Algeria and Tunisia, from 1948 in France, from 1953 in Morocco and from 1973 in Syria the truncated formulas were replaced with the full formulas.

To compute the Lambert Conic Near-Conformal the following formulas are used. First compute constants for the projection:

$n = f / (2 – f)$

$A = 1 / (6\ \rho_O\ \nu_O)$ where ρ_O and ν_O are computed as in table 3 in section 1.2 above.

$A' = a\ [\ 1 – n + 5\ (n^2 – n^3\ ) / 4 + 81\ (\ n^4 – n^5\ ) / 64]*\pi /180$

$B' = 3\ a\ [\ n – n^2 + 7\ (\ n^3 – n^4\ ) / 8 + 55\ n^5 / 64] / 2$

$C' = 15\ a\ [\ n^2 – n^3 + 3\ (\ n^4 – n^5\ ) / 4\ ] / 16$

$D' = 35\ a\ [\ n^3 – n^4 + 11\ n^5 / 16\ ] / 48$

$E' = 315\ a\ [\ n^4 – n^5\ ] / 512$

$r_O = k_O\ \nu_O / \tan \varphi_O$

$s_O = A'\ \varphi_O – B' \sin 2\varphi_O + C' \sin 4\varphi_O – D' \sin 6\varphi_O + E' \sin 8\varphi_O$

where in the first term φ_O is in degrees, in the other terms φ_O is in radians.

Then for the computation of easting and northing from latitude and longitude:

$s = A'\ \varphi – B' \sin 2\varphi + C' \sin 4\varphi – D' \sin 6\varphi + E' \sin 8\varphi$

where in the first term φ is in degrees, in the other terms φ is in radians.

$m = [s - s_O]$
$M = k_O (m + Am^3)$ (see footnote[4])
$r = r_O - M$
$\theta = (\lambda - \lambda_O) \sin \varphi_O$

and

$E = FE + r \sin\theta$
$N = FN + M + r \sin\theta \tan (\theta / 2)$

The reverse formulas for φ and λ from E and N are:

$\theta' = \text{atan} \{(E - FE) / [r_O - (N - FN)]\}$
$r' = \pm\{(E - FE)^2 + [r_O - (N - FN)]^2\}^{0.5}$, taking the sign of φ_O
$M' = r_O - r'$

If an exact solution is required, it is necessary to solve for m and φ using iteration of the two equations Firstly:

$m' = m' - [M' - k_O m' - k_O A (m')^3] / [- k_O - 3 k_O A (m')^2]$

using M' for m' in the first iteration. This will usually converge (to within 1mm) in a single iteration.
Then

$\varphi' = \varphi' + \{m' + s_O - [A' \varphi' (180/\pi) - B' \sin 2\varphi' + C' \sin 4\varphi' - D' \sin 6\varphi' + E' \sin 8\varphi']\}/A' (\pi/180)$

first using $\varphi' = \varphi_O + m'/A' (\pi/180)$.

However the following non-iterative solution is accurate to better than 0.001" (3mm) within 5 degrees latitude of the projection origin and should suffice for most purposes:

$m' = M' - [M' - k_O M' - k_O A (M')^3] / [- k_O - 3 k_O A (M')^2]$
$\varphi' = \varphi_O + m'/A' (\pi/180)$
$s' = A' \varphi' - B' \sin 2\varphi' + C' \sin 4\varphi' - D' \sin 6\varphi' + E' \sin 8\varphi'$
where in the first term φ' is in degrees, in the other terms φ' is in radians.
$ds' = A'(180/\pi) - 2B' \cos 2\varphi' + 4C' \cos 4\varphi' - 6D' \cos 6\varphi' + 8E' \cos 8\varphi'$
$\varphi = \varphi' - [(m' + s_O - s') / (-ds')]$ radians

Then after solution of φ using either method above

$\lambda = \lambda_O + \theta' / \sin \varphi_O$ where λ_O and λ are in radians

Example:
For Projected Coordinate Reference System: Deir ez Zor / Levant Zone

Parameters:

Ellipsoid:	Clarke 1880 (IGN)	a = 6378249.2 metres	1/f = 293.4660213
	then		n = 0.001706682563

Latitude of natural origin	φ_O	34°39'00"N		=	0.604756586 rad
Longitude of natural origin	λ_O	37°21'00"E		=	0.651880476 rad
Scale factor at natural origin	k_O	0.99962560			
False easting	FE	300000.00	metres		
False northing	FN	300000.00	metres		

Forward calculation for:

Latitude	φ	=	37°31'17.625"N	=	0.654874806 rad
Longitude	λ	=	34°08'11.291"E	=	0.595793792 rad

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[4] This is the term that is truncated to the third order. To be equivalent to the Lambert Conic Conformal (1SP) it would be $M = k_O (m + Am^3 + Bm^4 + Cm^5 + Dm^6)$. B, C and D are not detailed here.

first gives :

A	=	$4.1067494 * 10^{-15}$	A'	=	111131.8633
B'	=	16300.64407	C'	=	17.38751
D'	=	0.02308	E'	=	0.000033
s_O	=	3835482.233	r_O	=	9235264.405
s	=	4154101.458	m	=	318619.225
M	=	318632.72	r	=	8916631.685
θ	=	-0.03188875			

Then Easting E = 15707.96 metres (c.f. E = 15708.00 using full formulas)
Northing N = 623165.96 metres (c.f. N = 623167.20 using full formulas)

Reverse calculation for same easting and northing first gives:

θ' = -0.031888749
r' = 8916631.685
M' = 318632.717

Using the non-iterative solution:

m' = 318619.222
φ' = 0.654795830
s' = 4153599.259
ds' = 6358907.456

Then Latitude φ = 0.654874806 rad = 37°31'17.625"N
Longitude λ = 0.595793792 rad = 34°08'11.291"E

1.3.2 Krovak Oblique Conformal Conic

(EPSG dataset coordinate operation method code 9819)

The normal case of the Lambert Conformal conic is for the axis of the cone to be coincident with the minor axis of the ellipsoid, that is the axis of the cone is normal to the ellipsoid at a geographic pole. For the Oblique Conformal Conic the axis of the cone is normal to the ellipsoid at a defined location and its extension cuts the minor axis at a defined angle. The map projection method is similar in principle to the Oblique Mercator (see section 1.3.6). It is used in the Czech Republic and Slovakia under the name 'Krovak' projection, where like the Laborde oblique cylindrical projection in Madagascar (section 1.4.6.1) the rotation to north is made in spherical rather than plane coordinates. The geographic coordinates on the ellipsoid are first reduced to conformal coordinates on the conformal (Gaussian) sphere. These spherical coordinates are then rotated to north and the rotated spherical coordinates then projected onto the oblique cone and converted to grid coordinates. The pseudo standard parallel is defined on the conformal sphere after its rotation. It is then the parallel on this sphere at which the map projection is true to scale; on the ellipsoid it maps as a complex curve. A scale factor may be applied to the map projection to increase the useful area of coverage.

The defining parameters for the Krovak oblique conformal conic map projection are:

φ_C = latitude of projection centre, the point used as the origin of the conformal sphere
λ_O = longitude of origin
α_C = azimuth on conformal sphere of initial line passing through the projection centre
= co-latitude of the cone axis at point of intersection with the conformal sphere
φ_P = latitude of pseudo standard parallel
k_P = scale factor on pseudo standard parallel
FE = Easting at grid origin
FN = Northing at grid origin

The grid origin is the intersection on the conformal sphere of the pseudo-standard parallel with the longitude of origin.

From these the following constants for the projection may be calculated :

$$A = a (1 - e^2)^{0.5} / [1 - e^2 \sin^2 (\varphi_C)]$$
$$B = \{1 + [e^2 \cos^4\varphi_C / (1 - e^2)]\}^{0.5}$$
$$\gamma_O = \mathrm{asin}\,[\sin (\varphi_C) / B]$$
$$t_O = \tan(\pi / 4 + \gamma_O / 2) \,.\, [(1 + e \sin (\varphi_C)) / (1 - e \sin (\varphi_C))]^{e.B/2} / [\tan(\pi / 4 + \varphi_C/ 2)]^B$$
$$n = \sin (\varphi_P)$$
$$r_O = k_P A / \tan (\varphi_P)$$

To derive the projected 'Easting' and 'Northing' coordinates of a point with geographical coordinates (φ,λ) the formulas for the Krovak oblique conic conformal are:

Southing: $X = FN + r \cos \theta$
Westing: $Y = FE + r \sin \theta$

where

$$U = 2 (\mathrm{atan}\,\{ t_O \tan^B(\varphi/ 2 + \pi / 4) / [(1 + e \sin (\varphi)) / (1 - e \sin (\varphi))]^{e.B/2} \} - \pi / 4)$$
$$V = B (\lambda_O - \lambda)$$
$$S = \mathrm{asin}\,[\cos (\alpha_C) \sin (U) + \sin (\alpha_C) \cos (U) \cos (V)]$$
$$D = \mathrm{asin}\,[\cos (U) \sin (V) / \cos (S)]$$
$$\theta = n D$$
$$r = r_O \tan^n (\pi / 4 + \varphi_P/ 2) / \tan^n (S/2 + \pi / 4)$$

Note that the terms 'Easting' and 'Northing' here refer to the two map grid coordinates. Their actual geographic direction depends upon the azimuth of the centre line. Note also that the formula for D is satisfactory for the normal use of the projection within the pseudo-longitude range on the conformal sphere of ±90 degrees from the central line of the projection. Should there be a need to exceed this range (which is not necessary for application in the Czech and Slovak Republics) then for the calculation of D use:

$$\sin(D) = \cos(U) * \sin(V) / \cos(S)$$
$$\cos(D) = \{[\cos(\alpha_C)*\sin(S) - \sin(U)] / [\sin(\alpha_C)*\cos(S)]\}$$
$$D = \mathrm{atan2}(\sin D, \cos D)$$

The reverse formulas to derive the latitude and longitude of a point from its 'Easting' (Y) and 'Northing' (X) values are:

$$r' = [(Y - FE)^2 + (X - FN)^2]^{0.5}$$
$$\theta' = \mathrm{atan}\,[(Y - FE)/(X - FN)]$$
$$D' = \theta' / \sin (\varphi_P)$$
$$S' = 2\{\mathrm{atan}[(r_O / r')^{1/n} \tan(\pi / 4 + \varphi_P/ 2)] - \pi / 4\}$$
$$U' = \mathrm{asin}\,[\cos (\alpha_C) \sin (S') - \sin (\alpha_C) \cos (S') \cos (D')]$$
$$V' = \mathrm{asin}\,[\cos (S') \sin (D') / \cos (U')]$$

Latitude φ is found by iteration using U' as the value for φ_{j-1} in the first iteration

$$\varphi_j = 2 (\mathrm{atan}\,\{ t_O^{-1/B} \tan^{1/B} (U'/2 + \pi / 4) [(1 + e \sin (\varphi_{j-1}))/ (1 - e \sin (\varphi_{j-1}))]^{e/2} \} - \pi / 4)$$

3 iterations will usually suffice. Then:

$$\lambda = \lambda_O - V' / B$$

Example

For Projected Coordinate Reference System: S-JTSK (Ferro) / Krovak

N.B. Krovak projection uses Ferro as the prime meridian. This has a longitude with reference to Greenwich of 17 degrees 40 minutes West. To apply the formulas the defining longitudes must be corrected to the Greenwich meridian.

Parameters:

Ellipsoid:	Bessel 1841	a = 6377397.155metres	1/f = 299.15281
	then	e = 0.081696831	e^2 = 0.006674372

Latitude of projection centre	φ_C	49°30'00"N	=	0.863937979 rad
Longitude of origin	λ_O	42°30'00" East of Ferro		
Longitude of Ferro is		17°40'00" West of Greenwich		
λ_O relative to Greenwich:		24°50'00"	=	0.433423431 rad
Azimuth of initial line	α_C	30°17'17.3031"		
Latitude of pseudo standard parallel	φ_P	78°30'00"N		
Scale factor on pseudo standard parallel	k_P	0.9999		
Easting at grid origin	FE	0.00	metres	
Northing at grid origin	FN	0.00	metres	

Projection constants:

A	=	6380703.611	B	=	1.000597498
γ_O	=	0.863239103	t_O	=	1.003419164
n	=	0.979924705	r_O	=	1298039.005

Forward calculation for:

Latitude	φ	=	50°12'32.4416"N	=	0.876312566 rad
Longitude	λ	=	16°50'59.1790"E	=	0.294083999 rad

first gives :

U	=	0.875596949
V	=	0.139422687
S	=	1.386275049
D	=	0.506554623
θ	=	0.496385389
r	=	1194731.014

Then	'Northing'	X	=	1050538.643 metres
	'Easting'	Y	=	568990.997 metres

where 'Northing' increases southwards and 'Easting' increases westwards.

Reverse calculation for the same 'Northing' and 'Easting' gives

r'	=	1194731.014
θ'	=	0.496385389
D'	=	0.506554623
S'	=	1.386275049
U'	=	0.875596949
V'	=	0.139422687

Then by iteration

$\varphi 1 = 0.876310601$ rad
$\varphi 2 = 0.876312560$ rad
$\varphi 3 = 0.876312566$ rad

Latitude $\varphi = 0.876312566$ rad $= 50°12'32.4416"$N
Longitude $\lambda = 0.294083999$ rad $= 16°50'59.1790"$E of Greenwich

1.3.3 Mercator

(EPSG dataset coordinate operation method codes 9804 and 9805)

The Mercator map projection is a special limiting case of the Lambert Conic Conformal map projection with the equator as the single standard parallel. All other parallels of latitude are straight lines and the meridians are also straight lines at right angles to the equator, equally spaced. It is the basis for the transverse and oblique forms of the projection. It is little used for land mapping purposes but is in almost universal use for navigation charts. As well as being conformal, it has the particular property that straight lines drawn on it are lines of constant bearing. Thus navigators may derive their course from the angle the straight course line makes with the meridians.

In the few cases in which the Mercator projection is used for terrestrial applications or land mapping, such as in Indonesia prior to the introduction of the Universal Transverse Mercator, a scale factor may be applied to the projection. This has the same effect as choosing two standard parallels on which the true scale is maintained at equal north and south latitudes either side of the equator.

The formulas to derive projected Easting and Northing coordinates are:

For the two standard parallel case, k_O, the scale factor at the equator or natural origin, is first calculated from

$$k_O = \cos\varphi_1 / (1 - e^2 \sin^2\varphi_1)^{0.5}$$

where φ_1 is the absolute value of the first standard parallel (i.e. positive).

Then, for both one and two standard parallel cases,

$$E = FE + a\, k_O (\lambda - \lambda_O)$$
$$N = FN + a\, k_O \ln\{\tan(\pi/4 + \varphi/2)[(1 - e\sin\varphi)/(1 + e\sin\varphi)]^{(e/2)}\}$$

where symbols are as listed above and logarithms are natural.

The reverse formulas to derive latitude and longitude from E and N values are:

$$\varphi = \chi + (e^2/2 + 5e^4/24 + e^6/12 + 13e^8/360) \sin(2\chi)$$
$$+ (7e^4/48 + 29e^6/240 + 811e^8/11520) \sin(4\chi)$$
$$+ (7e^6/120 + 81e^8/1120) \sin(6\chi) + (4279e^8/161280) \sin(8\chi)$$

where $\chi = \pi/2 - 2 \operatorname{atan} t$

$t = B^{(FN-N)/(a.\ kO)}$
B = base of the natural logarithm, 2.7182818…
and for the 2 SP case, k_O is calculated as for the forward transformation above.

$$\lambda = [(E - FE)/ak_O] + \lambda_O$$

Note that in these formulas common to both 1SP and 2SP cases, the parameter *latitude of natural origin* (φ_O) is not used. However for the Merctor (1SP) method, for completeness in CRS labelling the EPSG dataset includes this parameter, which must have a value of zero.

Examples:

1. Mercator (with two standard parallels) (EPSG dataset coordinate operation method code 9805)

For Projected Coordinate Reference System: Pulkovo 1942 / Mercator Caspian Sea

Parameters:

Ellipsoid:	Krassowski 1940	$a = 6378245.0$ metres	$1/f = 298.3$
	then	$e = 0.08181333$	$e^2 = 0.00669342$

Latitude of 1st standard parallel	φ_1	42°00'00"N	=	0.73303829 rad
Longitude of natural origin	λ_O	51°00'00"E	=	0.89011792 rad
False easting	FE	0.00 metres		
False northing	FN	0.00 metres		

then scale factor at natural origin k_O (at latitude of natural origin at 0°N) = 0.744260894.

Forward calculation for:

Latitude	φ	=	53°00'00.00"N	=	0.9250245 rad
Longitude	λ	=	53°00'00.00"E	=	0.9250245 rad

gives	Easting	E	=	165704.29 metres
	Northing	N	=	5171848.07 metres

Reverse calculation for same easting and northing first gives:

$t = 0.336391288$

$\chi = 0.921795958$

Then	Latitude	φ	=	53°00'00.000"N
	Longitude	λ	=	53°00'00.000"E

2. Mercator (1SP) (EPSG dataset coordinate operation method code 9804)

For Projected Coordinate Reference System: Makassar / NEIEZ

Parameters:

Ellipsoid:	Bessel 1841	$a = 6377397.155$metres	$1/f = 299.15281$
	then	$e = 0.081696831$	

Latitude of natural origin	φ_O	0°00'00"N	=	0.0 rad
Longitude of natural origin	λ_O	110°00'00"E	=	1.91986218 rad
Scale factor at natural origin	k_O	0.997		
False easting	FE	3900000.00 metres		
False northing	FN	900000.00 metres		

Forward calculation for:

Latitude	φ	=	3°00'00.00"S	=	-0.05235988 rad
Longitude	λ	=	120°00'00.00"E	=	2.09439510 rad

gives	Easting	E	=	5009726.58 metres
	Northing	N	=	569150.82 metres

Reverse calculation for same easting and northing first gives:

t = 1.0534121
χ = -0.0520110

Then Latitude φ = 3°00'00.000"S
Longitude λ = 120°00'00.000"E

1.3.3.1 Mercator (Spherical)

(EPSG dataset coordinate operation method code 1026)

The formulas to derive projected Easting and Northing coordinates from spherical latitude φ and longitude λ are:

$$E = FE + R\,(\lambda - \lambda_O)$$
$$N = FN + R\ln[\tan(\pi/4 + \varphi/2)]$$

where λ_O is the longitude of natural origin and FE and FN are false easting and false nothing.

R is the radius of the sphere and will normally be one of the CRS parameters. If the figure of the earth used is an ellipsoid rather than a sphere then R should be calculated as the radius of the conformal sphere at the projection origin at latitude φ_O using the formula for R_C given in section 1.2, table 3. Note however that if applying spherical formula to ellipsoidal coordinates, the projection properties are not preserved.

If latitude φ = 90º, N is infinite. The above formula for N will fail near to the pole, and should not be used poleward of 88º.

The reverse formulas to derive latitude and longitude on the sphere from E and N values are:

$$D = -(N - FN) / R = (FN - N) / R$$
$$\varphi = \pi/2 - 2\,\mathrm{atan}(\mathbf{e}^D)$$ where **e**=base of natural logarithms, 2.7182818…
$$\lambda = [(E - FE)/R] + \lambda_O$$

Note that in these formulas, the parameter *latitude of natural origin* (φ_O) is not used. However for the Merctor (Spherical) method, for completeness in CRS labelling the EPSG dataset includes this parameter, which must have a value of zero.

Example

For Projected Coordinate Reference System: World Spherical Mercator

Parameters:

Sphere: R = 6371007.0 metres

Latitude of natural origin	φ_O	0°00'00.000"N	=	0.0 rad
Longitude of natural origin	λ_O	0°00'00.000"E	=	0.0 rad
False easting	FE	0.00	metres	
False northing	FN	0.00	metres	

Forward calculation for:

Latitude	φ	=	24°22'54.433"N	=	0.425542460 rad
Longitude	λ	=	100°20'00.000"W	=	–1.751147016 rad

whence

E = –11 156 569.90 m
N = 2 796 869.94 m

Reverse calculation for the same point (–11 156 569.90 m E, 2 796 869.94m N) first gives:

$D = -0.438999665$

Then	Latitude	φ	=	0.425542460 rad	=	24°22'54.433"N
	Longitude	λ	=	–1.751147016 rad	=	100°20'00.000"W

1.3.3.2 Popular Visualisation Pseudo Mercator

(EPSG dataset coordinate operation method code 1024)

This method is utilised by some popular web mapping and visualisation applications. It applies standard Mercator (Spherical) formulas (section 1.3.3.1 above) to ellipsoidal coordinates and the sphere radius is taken to be the semi-major axis of the ellipsoid. This approach only approximates to the more rigorous application of ellipsoidal formulas to ellipsoidal coordinates (as given in EPSG dataset coordinate operation method codes 9804 and 9805 in section 1.3.3 above). Unlike either the spherical or ellipsoidal Mercator projection methods, this method is not conformal: scale factor varies as a function of azimuth, which creates angular distortion. Despite angular distortion there is no convergence in the meridian.

The formulas to derive projected Easting and Northing coordinates from ellipsoidal latitude φ and longitude λ first derive the radius of the sphere (R) from:

$R = a$

Then applying spherical Mercator formulae:

$E = FE + R\,(\lambda - \lambda_O)$

$N = FN + R \ln[\tan(\pi/4 + \varphi/2)]$

where symbols are as listed in 1.3.3.1 above and logarithms are natural.

If latitude $\varphi = 90^\circ$, N is infinite. The above formula for N will fail near to the pole, and should not be used poleward of 88º.

The reverse formulas to derive latitude and longitude on the ellipsoid from E and N values are:

$D = -(N - FN) / R = (FN - N) / R$

$\varphi = \pi/2 - 2 \operatorname{atan}(\mathbf{e}^D)$ where **e**=base of natural logarithms, 2.7182818…

$\lambda = [(E - FE)/R] + \lambda_O$

q_α is the scale factor at a given azimuth α. It is a function of the radius of curvature at that azimuth, R', derived from:

$R' = \rho\,\nu / (\nu \cos^2\alpha + \rho \sin^2\alpha)$

$q_\alpha = R / (R' \cos\varphi)$

where ρ and ν are the radii of curvature of the ellipsoid at latitude φ in the plane of the meridian and perpendicular to the meridian respectively;

$\rho = a\,(1 - e^2) / (1 - e^2\sin^2\varphi)^{3/2}$

$\nu = a / (1 - e^2\sin^2\varphi)^{1/2}$

Then when the azimuth is 0º, 180º, 90º or 270º the scale factors in the meridian (h) and on the parallel (k) are:

$q_0 = q_{180} = h = R / (\rho \cos\varphi)$

$q_{90} = q_{270} = k = R / (\nu \cos\varphi)$

which demonstrates the non-conformallity of the Pseudo Mercator method.

Maximum angular distortion ω is a function of latitude and is found from:

$$\omega = 2 \operatorname{asin}\{[\mathrm{ABS}(h - k)] / (h + k)\}$$

Note that in these formulas, as with those of the Mercator (spherical) method above, the parameter *latitude of natural origin* (φ_O) is not used. However for completeness in CRS labelling the EPSG dataset includes this parameter, which must have a value of zero.

Example

For Projected Coordinate Reference System: WGS 84 / Pseudo-Mercator

Parameters:

Ellipsoid: WGS 84 a = 6378137.0 metres 1/f = 298.2572236

Latitude of natural origin	φ_O	0°00'00.000"N	=	0.0 rad
Longitude of natural origin	λ_O	0°00'00.000"E	=	0.0 rad
False easting	FE	0.00	metres	
False northing	FN	0.00	metres	

Forward calculation for the same coordinate values as used for the Mercator (Spherical) example in 1.3.3.1 above:

Latitude	φ	=	24°22'54.433"N	=	0.425542460 rad
Longitude	λ	=	100°20'00.000"W	=	–1.751147016 rad

R = 6378137.0

whence

E = –11 169 055.58 m

N = 2 800 000.00 m

and

h = 1.1034264

k = 1.0972914

ω = 0°19'10.01"

Reverse calculation for a point 10km north on the grid (–11 169 055.58 m E, 2 810 000.00m N) first gives:

D = –0.44056752

Then	Latitude	φ	=	0.426970023 rad	=	24°27'48.889"N
	Longitude	λ	=	–1.751147016 rad	=	100°20'00.000"W

In comparision, the same WGS 84 ellipsoidal coordinates when converted to the WGS 84 / World Mercator projected coordinate reference system (EPSG CRS code 3395) using the ellipsoidal Mercator (1SP) method described above results in a grid distance between the two points of 9944.4m, a scale difference of ~0.5%.

WGS 84		WGS 84 / Pseudo-Mercator		WGS 84 / World Mercator	
Latitude	Longitude	Easting	Northing	Easting	Northing
24°27'48.889"N	100°20'00.000"W	–11169055.58m	2810000.00m	–11169055.58m	2792311.49m
24°22'54.433"N	100°20'00.000"W	–11169055.58m	2800000.00m	–11169055.58m	2782367.06m
		0.00m	10000.00m	0.00m	9944.43m

1.3.4 Cassini-Soldner

(EPSG dataset coordinate operation method code 9806)

The Cassini-Soldner projection is the ellipsoidal version of the Cassini projection for the sphere. It is not conformal but as it is relatively simple to construct it was extensively used in the last century and is still useful for mapping areas with limited longitudinal extent. It has now largely been replaced by the conformal Transverse Mercator which it resembles. Like this, it has a straight central meridian along which the scale is true, all other meridians and parallels are curved, and the scale distortion increases rapidly with increasing distance from the central meridian.

The formulas to derive projected Easting and Northing coordinates are:

$$\text{Easting, } E = FE + \nu[A - TA^3/6 - (8 - T + 8C)TA^5/120]$$

$$\text{Northing, } N = FN + X$$

where $X = M - M_O + \nu\tan\varphi[A^2/2 + (5 - T + 6C)A^4/24]$
$A = (\lambda - \lambda_O)\cos\varphi$
$T = \tan^2\varphi$
$C = e^2 \cos^2\varphi/(1 - e^2)$
$\nu = a /(1 - e^2\sin^2\varphi)^{0.5}$

and M, the distance along the meridian from equator to latitude φ, is given by

$$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024 + ...)\sin2\varphi + (15e^4/256 + 45e^6/1024 + ...)\sin4\varphi - (35e^6/3072 + ...)\sin6\varphi + ...]$$

with φ in radians.

M_O is the value of M calculated for the latitude of the natural origin φ_O. This may not necessarily be chosen as the equator.

To compute latitude and longitude from Easting and Northing the reverse formulas are:

$$\varphi = \varphi_1 - (\nu_1\tan\varphi_1/\rho_1)[D^2/2 - (1 + 3T_1)D^4/24]$$

$$\lambda = \lambda_O + [D - T_1D^3/3 + (1 + 3T_1)T_1D^5/15]/\cos\varphi_1$$

where

$\nu_1 = a /(1 - e^2\sin^2\varphi_1)^{0.5}$
$\rho_1 = a(1 - e^2)/(1 - e^2\sin^2\varphi_1)^{1.5}$
φ_1 is the latitude of the point on the central meridian which has the same Northing as the point whose coordinates are sought, and is found from:

$$\varphi_1 = \mu_1 + (3e_1/2 - 27e_1^3/32 + ...)\sin2\mu_1 + (21e_1^2/16 - 55e_1^4/32 + ...)\sin4\mu_1 + (151e_1^3/96 + ...)\sin6\mu_1 + (1097e_1^4/512 - ....)\sin8\mu_1 + ...$$

where

$e_1 = [1 - (1 - e^2)^{0.5}]/[1 + (1 - e^2)^{0.5}]$
$\mu_1 = M_1/[a(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)]$
$M_1 = M_O + (N - FN)$
$T_1 = \tan^2\varphi_1$
$D = (E - FE)/\nu_1$

Example
For Projected Coordinate Reference System: Trinidad 1903 / Trinidad Grid

Parameters:

Ellipsoid:	Clarke 1858	a = 20926348 ft	=	31706587.88 Clarke's links
		b = 20855233 ft		
	then	1/f = 294.2606764		e^2 = 0.006785146

Latitude of natural origin	φ_O	10°26'30"N	=	0.182241463 rad
Longitude of natural origin	λ_O	61°20'00"W	=	-1.070468608 rad
False easting	FE	430000.00		Clarke's links
False northing	FN	325000.00		Clarke's links

Forward calculation for:

Latitude	φ	=	10°00'00.00"N	=	0.17453293 rad
Longitude	λ	=	62°00'00.00"W	=	-1.08210414 rad

first gives :

A	=	-0.01145876	C	=	0.00662550
T	=	0.03109120	M	=	5496860.24
ν	=	31709831.92	M_O	=	5739691.12

Then	Easting	E	=	66644.94 Clarke's links
	Northing	N	=	82536.22 Clarke's links

Reverse calculation for same easting and northing first gives:

e_1	=	0.00170207	D	=	-0.01145875
T_1	=	0.03109544	M_1	=	5497227.34
ν_1	=	31709832.34	μ_1	=	0.17367306
φ_1	=	0.17454458	ρ_1	=	31501122.40

Then	Latitude	φ	=	10°00'00.000"N
	Longitude	λ	=	62°00'00.000"W

1.3.4.1 Hyperbolic Cassini-Soldner
(EPSG dataset coordinate operation method code 9833)

The grid for the island of Vanua Levu, Fiji, uses a modified form of the standard Cassini-Soldner projection known as the Hyperbolic Cassini-Soldner.

Easting is calculated as for the standard Cassini-Soldner above. The standard Cassini-Soldner formula to derive projected Northing is modified to:

$$\text{Northing, } N = FN + X - (X^3/6\rho\nu)$$

where $\rho = a(1 - e^2)/(1 - e^2\sin^2\varphi)^{1.5}$ and X and ν are as in the standard Cassini-Soldner above.

For the reverse calculation of latitude and longitude from easting and northing the standard Cassini-Soldner formula given in the previous section need to be modified to account for the hyperbolic correction factor $(X^3/6\rho\nu)$. *Specifically for the Fiji Vanua Levu grid* the following may be used. The standard Cassini-Soldner formula given in the previous section are used except that the equation for M_1 is modified to

$$M_1 = M_O + (N - FN) + q$$

where

$\varphi_1' = \varphi_O + (N - FN)/315320$
$\rho_1' = a(1 - e^2)/(1 - e^2\sin^2\varphi_1')^{1.5}$
$\nu_1' = a /(1 - e^2\sin^2\varphi_1')^{0.5}$
$q' = (N - FN)^3 / 6 \rho_1' \nu_1'$
$q = (N - FN + q')^3 / 6 \rho_1' \nu_1'$

Example
For Projected Coordinate Reference System: Vanua Levu 1915 / Vanua Levu Grid

Parameters:

Ellipsoid:	Clarke 1880		a = 20926202 ft	=	317063.667 chains
			b = 20854895 ft		
		then	1/f = 293.4663077		e^2 = 0.006803481

Latitude of natural origin	φ_O	16°15'00"S	=	-0.283616003 rad
Longitude of natural origin	λ_O	179°20'00"E	=	3.129957125 rad
False easting	FE	12513.318	chains	
False northing	FN	16628.885	chains	

Forward calculation for:

Latitude	φ	=	16°50'29.2435"S	=	-0.293938867 rad
Longitude	λ	=	179°59'39.6115"E	=	3.141493807 rad

first gives :

A	=	0.011041875	C	=	0.006275088
T	=	0.091631819	M	=	-92590.02
ν	=	317154.24	M_O	=	-89336.59
ρ	=	315176.48	X	=	-3259.28

Then	Easting	E	=	16015.2890 chains
	Northing	N	=	13369.6601 chains

Reverse calculation for same easting and northing first gives:`

φ_1'	=	0.293952249	q'	=	-0.058
ν_1'	=	317154.25	q	=	-0.058
ρ_1'		315176.50			
e_1	=	0.001706681	D	=	0.011041854
T_1	=	0.091644092	M_1	=	-92595.87
ν_1	=	317154.25	μ_1	=	-0.292540098
ρ_1	=	315176.51	φ_1	=	-0.293957437

Then	Latitude	φ	=	16°50'29.2435"S
	Longitude	λ	=	179°59'39.6115"E

1.3.5 Transverse Mercator

1.3.5.1 General Case
(EPSG dataset coordinate operation method code 9807)

The Transverse Mercator projection in its various forms is the most widely used projected coordinate system for world topographical and offshore mapping. All versions have the same basic characteristics and formulas. The differences which distinguish the different forms of the projection which are applied in different countries arise from variations in the choice of values for the coordinate conversion parameters, namely the latitude of the natural origin, the longitude of the natural origin (central meridian), the scale factor at the natural origin (on the central meridian), and the values of False Easting and False Northing, which embody the units of measurement, given to the origin. Additionally there are variations in the width of the longitudinal zones for the projections used in different territories.

The following table indicates the variations in the coordinate conversion parameters which distinguish the different forms of the Transverse Mercator projection and are used in the EPSG dataset Transverse Mercator map projection operations:

TABLE 4
Transverse Mercator

Coordinate Operation Method Name	Areas used	Central meridian	Latitude of natural origin	CM Scale Factor	Zone width	False Easting	False Northing
Transverse Mercator	Various, world wide	Various	Various	Various	Usually less than 6°	Various	Various
Transverse Mercator south oriented	Southern Africa	2° intervals E of 11°E	0°	1.000000	2°	0m	0m
UTM North hemisphere	World wide equator to 84 °N	6° intervals E & W of 3°E & W	Always 0°	Always 0.9996	Always 6°	500000 m	0m
UTM South hemisphere	World wide north of 80°S to equator	6° intervals E & W of 3°E & W	Always 0°	Always 0.9996	Always 6°	500000 m	10000000 m
Gauss-Kruger	Former USSR, Yugoslavia, Germany, S. America, China	Various, according to area of cover	Usually 0 °	Usually 1.000000	Usually less than 6°, often less than 4°	Various but often 500000 prefixed by zone number	Various
Gauss Boaga	Italy	Various	Various	0.9996	6°	Various	0m

The most familiar and commonly used Transverse Mercator in the oil industry is the Universal Transverse Mercator (UTM) whose natural origin lies on the equator. However, some territories use a Transverse Mercator with a natural origin at a latitude of natural origin closer to that territory.

In the EPSG dataset the coordinate conversion method is considered to be the same for all forms of the Transverse Mercator projection. The formulas to derive the projected Easting and Northing coordinates for the normal case (EPSG dataset coordinate operation method code 9807) are in the form of a series as follows:

$$\text{Easting, } E = FE + k_O\nu[A + (1 - T + C)A^3/6 + (5 - 18T + T^2 + 72C - 58e'^2)A^5/120]$$

$$\text{Northing, } N = FN + k_O\{M - M_O + \nu\tan\varphi[A^2/2 + (5 - T + 9C + 4C^2)A^4/24 +$$

$$(61 - 58T + T^2 + 600C - 330e'^{\,2})A^6/720]\}$$

where $T = \tan^2\varphi$

$C = e^2 \cos^2\varphi / (1 - e^2)$

$A = (\lambda - \lambda_O)\cos\varphi$, with λ and λ_O in radians

$\nu = a / (1 - e^2\sin^2\varphi)^{0.5}$

$$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024+....)\sin 2\varphi + (15e^4/256 + 45e^6/1024 +.....)\sin 4\varphi - (35e^6/3072 + ...)\sin 6\varphi + ...]$$

with φ in radians and M_O for φ_O, the latitude of the origin, derived in the same way.

The reverse formulas to convert Easting and Northing projected coordinates to latitude and longitude are:

$$\varphi = \varphi_1 - (\nu_1 \tan\varphi_1/\rho_1)[D^2/2 - (5 + 3T_1 + 10C_1 - 4C_1^{\,2} - 9e'^2)D^4/24 + (61 + 90T_1 + 298C_1 + 45T_1^{\,2} - 252e'^2 - 3C_1^{\,2})D^6/720]$$

$$\lambda = \lambda_O + [D - (1 + 2T_1 + C_1)D^3/6 + (5 - 2C_1 + 28T_1 - 3C_1^{\,2} + 8e'^2 + 24T_1^{\,2})D^5/120] / \cos\varphi_1$$

where

$\nu_1 = a /(1 - e^2\sin^2\varphi_1)^{0.5}$

$\rho_1 = a(1 - e^2)/(1 - e^2\sin^2\varphi_1)^{1.5}$

φ_1 may be found as for the Cassini projection from:

$$\varphi_1 = \mu_1 + (3e_1/2 - 27e_1^{\,3}/32 +.....)\sin 2\mu_1 + (21e_1^{\,2}/16 - 55e_1^{\,4}/32 + ...)\sin 4\mu_1 + (151e_1^{\,3}/96 +.....)\sin 6\mu_1 + (1097e_1^{\,4}/512 - ....)\sin 8\mu_1 + ...$$

and where

$e_1 = [1 - (1 - e^2)^{0.5}]/[1 + (1 - e^2)^{0.5}]$

$\mu_1 = M_1/[a(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)]$

$M_1 = M_O + (N - FN)/k_0$

$T_1 = \tan^2\varphi_1$

$C_1 = e'^2\cos^2\varphi_1$

$e'^2 = e^2 /(1 - e^2)$

$D = (E - FE)/(\nu_1 k_O)$

For areas south of the equator the value of latitude φ will be negative and the formulas above, to compute the E and N, will automatically result in the correct values. Note that the false northings of the origin, if the equator, will need to be large to avoid negative northings and for the UTM projection is in fact 10,000,000m. Alternatively, as in the case of Argentina's Transverse Mercator (Gauss-Kruger) zones, the origin is at the south pole with a northings of zero. However each zone central meridian takes a false easting of 500000m prefixed by an identifying zone number. This ensures that instead of points in different zones having the same eastings, every point in the country, irrespective of its projection zone, will have a unique set of projected system coordinates. Strict application of the above formulas, with south latitudes negative, will result in the derivation of the correct Eastings and Northings.

Similarly, in applying the reverse formulas to determine a latitude south of the equator, a negative sign for φ results from a negative φ_1 which in turn results from a negative M_1.

Example
For Projected Coordinate Reference System OSGB 1936 / British National Grid

Parameters:

Ellipsoid:	Airy 1830	a = 6377563.396 metres	1/f = 299.32496
	then	e^2 = 0.00667054	e'^2 = 0.00671534

Latitude of natural origin	φ_O	49°00'00"N	=	0.85521133 rad
Longitude of natural origin	λ_O	2°00'00"W	=	-0.03490659 rad
Scale factor at natural origin	k_O	0.9996012717		
False easting	FE	400000.00	metres	
False northing	FN	-100000.00	metres	

Forward calculation for:

Latitude	φ	=	50°30'00.00"N	=	0.88139127 rad
Longitude	λ	=	00°30'00.00"E	=	0.00872665 rad

first gives :

A	=	0.02775415	C	=	0.00271699
T	=	1.47160434	M	=	5596050.46
ν	=	6390266.03	M_O	=	5429228.60

Then Easting E = 577274.99 metres
Northing N = 69740.50 metres

Reverse calculation for same easting and northing first gives:

e_1	=	0.00167322	μ_1	=	0.87939562
M_1	=	5599036.80	ν_1	=	6390275.88
φ_1	=	0.88185987	D	=	0.02775243
ρ_1	=	6372980.21	C_1	=	0.00271391
T_1	=	1.47441726			

Then Latitude φ = 50°30'00.000"N
Longitude λ = 00°30'00.000"E

1.3.5.2 Transverse Mercator Zoned Grid System
(EPSG dataset coordinate operation method code 9824)

When the growth in distortion away from the projection origin is of concern, a projected coordinate reference system cannot be used far from its origin. A means of creating a grid system over a large area but also limiting distortion is to have several grid zones with most defining parameters being made common. Coordinates throughout the system are repeated in each zone. To make coordinates unambiguous the easting is prefixed by the relevant zone number. This procedure was adopted by German mapping in the 1930's through the Gauss-Kruger systems and later by American military mapping through the Universal Transverse Mercator (or UTM) grid system. (Note: subsequent civilian adoption of the systems usually ignores the zone prefix to easting. Where this is the case the formulas below do not apply: use the standard TM formula separately for each zone).

The parameter Longitude of natural origin (λ_O) is changed from being a defining parameter to a derived parameter, replaced by two other defining parameters, the Initial Longitude (the western limit of zone 1) (λ_I) and the Zone Width (W). Each of the remaining four Transverse Mercator defining parameters – Latitude of natural origin, Scale factor at natural origin, False easting and False northing – have the same parameter values in every zone.

The standard Transverse Mercator formulas above are modified as follows:

Zone number, Z, = INT(($\lambda + \lambda_I$ + W) / W) with λ, λ_I and W in degrees.
where λ_I is the Initial Longitude of the zoned grid system
and W is the width of each zone of the zoned grid system.
If $\lambda < 0$, $\lambda = (\lambda + 360)$ degrees.

Then,

$$\lambda_O = [Z\ W] - [\lambda_I + (W/2)]$$

For the forward calculation,

$$\text{Easting, } E = Z*10^6 + FE + k_O.\nu\ [A + (1 - T + C)A^3/6 + (5 - 18T + T^2 + 72C - 58e'^2)A^5/120]$$

and in the reverse calculation for longitude,

$$D = (E - [FE + Z*10^6])/(\nu_1\ k_O)$$

1.3.5.3 <u>Transverse Mercator (South Orientated)</u>

(EPSG dataset coordinate operation method code 9808)

For the mapping of southern Africa a south oriented Transverse Mercator map projection method is used. Here the coordinate axes are called Westings and Southings and increment to the West and South from the origin respectively. See Figure 3 for a diagrammatic illustration. The standard Transverse Mercator above formulas need to be modified to cope with this arrangement with

$$\text{Westing, } W = FE - k_O\ \nu[A + (1 - T + C)A^3/6 + (5 - 18T + T^2 + 72C - 58e'^2)A^5/120]$$

$$\text{Southing, } S = FN - k_O\{M - M_O + \nu\tan\varphi[A^2/2 + (5 - T + 9C + 4C^2)A^4/24 + (61 - 58T + T^2 + 600C - 330e'^2)A^6/720]\}$$

In these formulas the terms FE and FN retain their definition, i.e. in the Transverse Mercator (South Orientated) method they increase the Westing and Southing value at the natural origin. In this method they are effectively false westing (FW) and false southing (FS) respectively.

For the reverse formulas, those for the standard Transverse Mercator above apply, with the exception that:

$$M_1 = M_O - (S - FN)/k_O$$

and $D = -(W - FE)/(\nu_1\ k_O)$, with $\nu_1 = \nu$ for φ_1

1.3.6 <u>Oblique Mercator and Hotine Oblique Mercator</u>

(EPSG datset coordinate operation method codes 9815 and 9812).

It has been noted that the Transverse Mercator map projection method is employed for the topographical mapping of longitudinal bands of territories, limiting the amount of scale distortion by limiting the extent of the projection either side of the central meridian. Sometimes the shape, general trend and extent of some countries makes it preferable to apply a single zone of the same kind of projection but with its central line aligned with the trend of the territory concerned rather than with a meridian. So, instead of a meridian forming this true scale central line for one of the various forms of Transverse Mercator, or the equator forming the line for the Mercator, a line with a particular azimuth traversing the territory is chosen and the same principles of construction are applied to derive what is now an Oblique Mercator. Such a single zone projection suits areas which have a large extent in one direction but limited extent in the perpendicular direction and whose trend is oblique to the bisecting meridian - such as East and West Malaysia and the Alaskan panhandle. It was originally applied at the beginning of the 20th century by Rosenmund to the mapping of Switzerland, and in the 1970's adopted in Hungary. The projection's initial line may be selected as a line with a particular azimuth through a single point, normally at the centre of the mapped area, or as the geodesic line (the shortest line between two points on the ellipsoid) between two selected points.

OGP identifies two forms of the oblique Mercator projection, differentiated only by the point at which false grid coordinates are defined. If the false grid coordinates are defined at the intersection of the initial line and the aposphere, that is at the natural origin of the coordinate system, the map projection method is known as the Hotine Oblique Mercator (EPSG dataset coordinate operation method code 9812). If the false grid coordinates are defined at the projection centre the projection method is known as the Oblique Mercator (EPSG dataset coordinate operation method code 9815).

Hotine projected the ellipsoid conformally onto a sphere of constant total curvature, called the 'aposphere', before projection onto the plane and then rotation of the grid to north. This projection is sometimes referred to as the Rectified Skew Orthomorphic. Formulas, involving hyperbolic functions, were derived by Hotine. Snyder adapted these formulas to use exponential functions, thus avoiding use of Hotine's hyperbolic expressions. Alternative formulas derived by projecting the ellipsoid onto the 'conformal' sphere give identical results within the practical limits of the use of the formulas.

Snyder describes a variation of the Hotine Oblique Mercator where the initial line is defined by two points through which it passes. The latter approach is not currently followed in the EPSG dataset. It has been applied to mapping space imagery or, more frequently, for applying a geographical graticule to the imagery. However, the repeated path of the imaging satellite does not actually follow the centre lines of successive oblique cylindrical projections so a projection was derived whose centre line does follow the satellite path. This is known as the Space Oblique Mercator Projection and although it closely resembles an oblique cylindrical it is not quite conformal and has no application other than for space imagery.

The Oblique Mercator co-ordinate system is defined by:

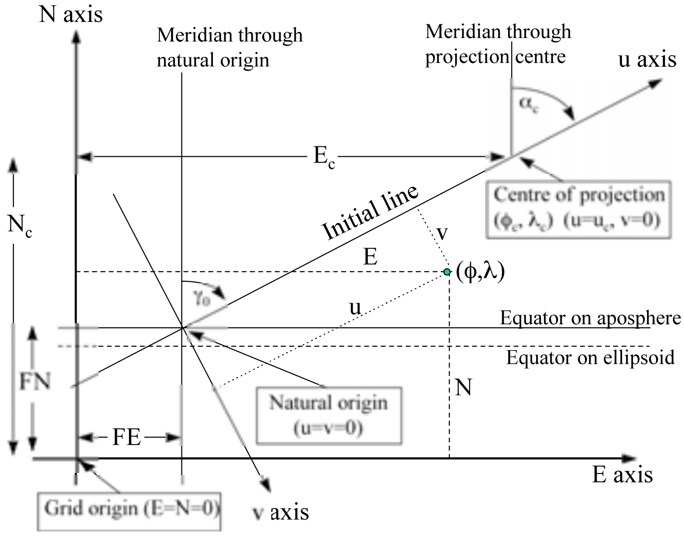

Figure 7. Key Diagram for Oblique Mercator Projection

The initial line central to the map area of given azimuth α_C passes through a defined centre of the projection (φ_C, λ_C). The point where the projection of this line cuts the equator on the aposphere is the origin of the (u , v) co-ordinate system. The u axis is along the initial line and the v axis is perpendicular to (90° clockwise from) this line.

In applying the formulas for the (Hotine) Oblique Mercator the first set of co-ordinates computed are referred to the (u, v) co-ordinate axes defined with respect to the initial line. These co-ordinates are then 'rectified' to the usual Easting and Northing by applying an orthogonal conversion. Hence the alternative name as the Rectified Skew Orthomorphic. The angle from rectified to skewed grid may be defined such that grid north coincides with true north at the natural origin of the projection, that is where the initial line of the projection intersects equator on the aposphere. In some circumstances, particularly where the projection is used in non-equatorial areas such as the Alaskan panhandle, the angle from rectified to skewed grid is defined to be identical to the azimuth of the initial line at the projection centre; this results in grid and true north coinciding at the projection centre rather than at the natural origin.

To ensure that all co-ordinates in the map area have positive grid values, false co-ordinates are applied. These may be given values (E_C , N_C) if applied at the projection centre [EPSG dataset Oblique Mercator method] or be applied as false easting (FE) and false northing (FN) at the natural origin [EPSG dataset Hotine Oblique Mercator method].

The formulas can be used for the following cases:

Alaska State Plane Zone 1
Hungary EOV
East and West Malaysia Rectified Skew Orthomorphic grids
Swiss Cylindrical projection

The Swiss and Hungarian systems are a special case where the azimuth of the line through the projection centre is 90 degrees.
The formulas may also be used as an approximation to the Laborde Grid for Madagscar (see following section).

Specific references for the formulas originally used in the individual cases of these projections are:
Switzerland: "Die Änderung des Projektionssystems der schweizerischen Landesvermessung." M. Rosenmund 1903. Also "Die projecktionen der Schweizerischen Plan und Kartenwerke." J. Bollinger 1967.
Madagascar: "La nouvelle projection du Service Geographique de Madagascar". J. Laborde 1928.
Malaysia: Series of Articles in numbers 62-66 of the Empire Survey Review of 1946 and 1947 by M. Hotine.

The defining parameters for the [Hotine] Oblique Mercator projection are:
φ_C = latitude of the projection centre
λ_C = longitude of the projection centre
α_C = azimuth (true) of the initial line passing through the projection centre
γ_C = angle from the rectified grid to the skew (oblique) grid
k_C = scale factor on the initial line of the projection
and either
for the Oblique Mercator:
E_C = False Easting at the centre of projection
N_C = False Northing at the centre of projection
or for the Hotine Oblique Mercator:
FE = False Easting at the natural origin
FN = False Northing at the natural origin

From these defining parameters the following constants for the map projection may be calculated for both the Hotine Oblique Mercator and Oblique Mercator methods:

B = $\{1 + [e^2 \cos^4\varphi_C / (1 - e^2)]\}^{0.5}$
A = $a\, B\, k_C\, (1 - e^2)^{0.5} / (1 - e^2 \sin^2 \varphi_C)$
t_O = $\tan(\pi/4 - \varphi_C / 2) / [(1 - e \sin \varphi_C) / (1 + e \sin \varphi_C)]^{e/2}$
D = $B\, (1 - e^2)^{0.5} / [\cos \varphi_C\, (1 - e^2 \sin^2 \varphi_C)^{0.5}]$
To avoid problems with computation of F, if D < 1 make $D^2 = 1$
F = $D + (D^2 - 1)^{0.5}\, .\, \mathrm{SIGN}(\varphi_C)$
H = $F\, t_O^B$
G = $(F - 1/F) / 2$
γ_O = $\mathrm{asin}[\sin(\alpha_C) / D]$
λ_O = $\lambda_C - [\mathrm{asin}(G \tan\gamma_O)] / B$

Then for the Oblique Mercator method only, two further constants for the map projection, the (u_C , v_C) co-ordinates for the centre point (φ_C , λ_C), are calculated from:
v_C = 0

In general
u_C = $(A / B)\, \mathrm{atan}[(D^2 - 1)^{0.5} / \cos(\alpha_C)] * \mathrm{SIGN}(\varphi_C)$

but for the special cases where α_c = 90 degrees (e.g. Hungary, Switzerland) then
u_C = $A\, (\lambda_C - \lambda_O)$

Forward case: To compute (E,N) from a given (φ,λ), for both the Hotine Oblique Mercator method and the Oblique Mercator method:

t = $\tan(\pi / 4 - \varphi / 2) / [(1 - e \sin \varphi) / (1 + e \sin \varphi)]^{e/2}$
Q = H / t^B
S = $(Q - 1/Q) / 2$
T = $(Q + 1/Q) / 2$
V = $\sin(B\, (\lambda - \lambda_O))$
U = $(- V \cos(\gamma_O) + S \sin(\gamma_O)) / T$
v = $A \ln[(1 - U) / (1 + U)] / (2 B)$

Then either
(a) for the Hotine Oblique Mercator (where the FE and FN values have been specified with respect to the natural origin of the (u , v) axes):
u = $A\, \mathrm{atan}\{(S \cos\gamma_O + V \sin\gamma_O) / \cos[B\, (\lambda - \lambda_O)]\} / B$

The rectified skew co-ordinates are then derived from:
E = $v \cos(\gamma_C) + u \sin(\gamma_C) + FE$
N = $u \cos(\gamma_C) - v \sin(\gamma_C) + FN$

or
(b) for the Oblique Mercator (where the false easting and northing values (E_C , N_C) have been specified with respect to the centre of the projection (φ_C , λ_C) then :
u = $(A\, \mathrm{atan}\{(S \cos\gamma_O + V \sin\gamma_O) / \cos[B\, (\lambda - \lambda_O)]\} / B) - (\mathrm{ABS}(u_C) * \mathrm{SIGN}(\varphi_C))$

The rectified skew co-ordinates are then derived from:
E = $v \cos(\gamma_C) + u \sin(\gamma_C) + E_C$
N = $u \cos(\gamma_C) - v \sin(\gamma_C) + N_C$

Reverse case: To compute (φ,λ) from a given (E,N) :

For the Hotine Oblique Mercator:

$v' = (E - FE)\cos(\gamma_C) - (N - FN)\sin(\gamma_C)$

$u' = (N - FN)\cos(\gamma_C) + (E - FE)\sin(\gamma_C)$

or for the Oblique Mercator:

$v' = (E - E_C)\cos(\gamma_C) - (N - N_C)\sin(\gamma_C)$

$u' = (N - N_C)\cos(\gamma_C) + (E - E_C)\sin(\gamma_C) + (ABS(u_C) * SIGN(\varphi_C)$

then for both cases:

$Q' = \mathbf{e}^{-(B v' / A)}$ where $\mathbf{e}$ is the base of natural logarithms.

$S' = (Q' - 1 / Q') / 2$

$T' = (Q' + 1 / Q') / 2$

$V' = \sin (B u' / A)$

$U' = (V' \cos(\gamma_O) + S' \sin(\gamma_O)) / T'$

$t' = \{H / [(1 + U') / (1 - U')]^{0.5}\}^{1 / B}$

$\chi = \pi / 2 - 2 \operatorname{atan}(t')$

$$\varphi = \chi + \sin(2\chi) (e^2 / 2 + 5 e^4 / 24 + e^6 / 12 + 13 e^8 / 360) + \sin(4\chi) (7 e^4 /48 + 29 e^6 / 240 + 811 e^8 / 11520) + \sin(6\chi) (7 e^6 / 120 + 81 e^8 / 1120) + \sin(8\chi) (4279 e^8 / 161280)$$

$\lambda = \lambda_O - \operatorname{atan} [(S' \cos\gamma_O - V' \sin\gamma_O) / \cos(B u' / A)] / B$

Examples:

For Projected Coordinate Reference System Timbalai 1948 / R.S.O. Borneo (m) using the Oblique Mercator method: (EPSG dataset coordinate operation method code 9815).

Parameters:

Ellipsoid: Everest 1830 (1967 Definition) a = 6377298.556 metres 1/f = 300.8017
then e = 0.081472981 e^2 = 0.006637847

Latitude of projection centre	φ_C	4°00'00"N	=	0.069813170 rad
Longitude of projection centre	λ_C	115°00'00"E	=	2.007128640 rad
Azimuth of initial line	α_C	53°18'56.9537"	=	0.930536611 rad
Angle from Rectified to Skew Grid	γ_C	53°07'48.3685"	=	0.927295218 rad
Scale factor on initial line	k_C	0.99984		
Easting at projection centre	E_C	590476.87	metres	
Northings at projection centre	N_C	442857.65	metres	

Constants for the map projection:

B	=	1.003303209	F	=	1.072121256
A	=	6376278.686	H	=	1.000002991
t_O	=	0.932946976	γ_O	=	0.927295218
D	=	1.002425787	λ_O	=	1.914373469
D^2	=	1.004857458			
u_c	=	738096.09	v_c	=	0.00

Forward calculation for:

Latitude	φ	=	5°23'14.1129"N	=	0.094025313 rad
Longitude	λ	=	115°48'19.8196"E	=	2.021187362 rad

first gives :

t	=	0.910700729	Q	=	1.098398182
S	=	0.093990763	T	=	1.004407419
V	=	0.106961709	U	=	0.010967247
v	=	-69702.787	u	=	163238.163

Then	Easting	E	=	679245.73 metres
	Northing	N	=	596562.78 metres

Reverse calculation for same easting and northing first gives:

v'	=	-69702.787	u'	=	901334.257
Q'	=	1.011028053			
S'	=	0.010967907	T'	=	1.000060146
V'	=	0.141349378	U'	=	0.093578324
t'	=	0.910700729	χ	=	0.093404829

Then	Latitude	φ	=	5°23'14.113"N
	Longitude	λ	=	115°48'19.820"E

If the same projection is defined using the Hotine Oblique Mercator method then:

False easting	FE	=	0.0 metres
False northing	FN	=	0.0 metres
Then	u	=	901334.257

and all other values are as for the Oblique Mercator method.

1.3.6.1 Laborde projection for Madagascar

(EPSG datset coordinate operation method code 9813).

For the mapping of Madagascar, Laborde developed a grid based on an oblique cylindrical conformal projection similar to the Oblique Mercator. Like Hotine's development for the Oblique Mercator, Laborde used a triple projection technique to map the ellipsoid to the plane. But in the Laborde projection the rotation to north is made on the intermediate conformal sphere rather than in the projection plane. Within 450 kilometres of the projection origin near Antananarivo, Laborde's formulas can be approximated to better than 2cm by the Oblique Mercator method described above, which is satisfactory for most purposes. However, beyond these limits, particularly in the direction along the initial line, results from the Oblique Mercator formulae diverge rapidly from those given by Laborde's formulas, so that at 600 kilometres from the origin along the initial line the Oblique Mercator approximates Laborde's formulas to no better than 1 metre.

The defining parameters for the Laborde Madagascar projection are:

φ_C = latitude of the projection centre
λ_C = longitude of the projection centre
α_C = azimuth (true) of the initial line passing through the projection centre
k_C = scale factor on the initial line of the projection
FE = False Easting at the natural origin
FN = False Northing at the natural origin

(Note: if the Oblique Mercator method is used as an approximation to the Laborde Madagascar, the additional parameter required by that method, the angle from the rectified grid to the skew (oblique) grid γ_C, takes the same value as the azimuth of the initial line passing through the projection centre, i.e. $\gamma_C = \alpha_C$)

All angular units should be converted to radians prior to use and all longitudes reduced to the Paris Meridan using the Paris Longitude of 2.5969212963 grads (2° 20' 14.025"E) east of Greenwich.

From these defining parameters the following constants for the map projection may be calculated:

$B = \{1+[e^2 \cos^4\varphi_C]/(1- e^2)\}^{0.5}$
$\varphi_s = \mathrm{asin}[\sin\varphi_C / B]$
$R = a\ k_C \{(1-e^2)^{0.5} / [1-e^2 \sin^2\varphi_C]\}$
$C = \ln[\tan(\pi/4+\varphi_s /2)] - B. \ln\{\tan(\pi/4+\varphi_c /2) ([1 - e \sin\varphi_c]/[1+e \sin\varphi_c])^{(e/2)}\}$

Forward case: To compute (E,N) from a given (φ,λ)
$L = B.(\lambda-\lambda_C)$
$q = C + B . \ln\{\tan(\pi/4+\varphi/2) ([1-e \sin\varphi] / [1+e \sin\varphi])^{(e/2)}\}$
$P = 2.\mathrm{atan}(\mathbf{e}^{q}) - \pi/2$ where $\mathbf{e}$ is the base of natural logarithms.
$U = \cos P.\cos L.\cos\varphi_s + \sin P.\sin\varphi_s$
$V = \cos P.\cos L.\sin\varphi_s - \sin P.\cos\varphi_s$
$W = \cos P.\sin L$
$d = (U^2+V^2)^{0.5}$
if d <> 0 then L' = 2.atan(V/(U+d)) and P' = atan(W/d)
if d = 0 then L' = 0 and P' = sign(W). $\pi/2$
$H = -L' + i.\ln(\tan(\pi/4+P'/2))$ where $i^2 = -1$
$G = (1 - \cos(2.\alpha_C) + i.\sin(2.\alpha_C))/12$
$E = E_C + R . \mathrm{IMAGINARY}(H+G.H^3)$
$N = N_C + R . \mathrm{REAL}(H + G.H^3)$

Reverse case: To compute (φ,λ) from a given (E,N):
$G = (1-\cos(2.\alpha_C) + i.\sin(2.\alpha_C))/12$ where $i^2 = -1$

To solve for Latitude and Longitude, a re-iterative solution is required, where the first two elements are
$H_0 = (N-FN)/R + i.(E-FE)/R$ ie k = 0
$H_1 = H_0/(H_0 + G.H_0^3)$, i.e. k = 1,
and in subsequent reiterations, k increments by 1
$H_{k+1} = (H_0+2.G.H_k^3)/(3.G.H_k^2+1)$
Re-iterate until $\mathrm{ABSOLUTE}(\mathrm{REAL}([H_0-H_k-G.H_k^3])) < 1E\text{-}11$

$L' = -1.\mathrm{REAL}(H_k)$
$P' = 2.\mathrm{atan}(\mathbf{e}^{\mathrm{IMAGINARY}(Hk)}) - \pi/2$ where $\mathbf{e}$ is the base of natural logarithms.
$U' = \cos P'.\cos L'.\cos\varphi_s + \cos P'.\sin L'.\sin\varphi_s$
$V' = \sin P'$
$W' = \cos P'.\cos L'.\sin\varphi_s - \cos P'.\sin L'.\cos\varphi_s$
$d = (U'^2+ V'^2)^{0.5}$
if d <> 0 then L = 2 atan[V'/(U'+d)] and P = atan(W'/d)
if d = 0 then L = 0 and P = SIGN(W') . $\pi/2$
$\lambda = \lambda_C + (L/B)$

$q' = \{\ln[\tan(\pi/4+P/2)] - C\}/B$
The final solution for latitude requires a second re-iterative process, where the first element is
$\varphi'_0 = 2.\mathrm{atan}(\mathbf{e}^{q'}) - \pi/2$ where $\mathbf{e}$ is the base of natural logarithms.
And the subsequent elements are
$\varphi'_k = 2.\mathrm{atan}\{(\{1+e.\sin[\varphi'_{k-1}]\} / \{1-e.\sin[\varphi'_{k-1}]\})^{(e/2)}.\ \mathbf{e}^{q'}\} - \pi/2$ for $k = 1 \rightarrow$
Iterate until $\mathrm{ABSOLUTE}(\varphi'_k - \varphi'_{k-1}) < 1E\text{-}11$
$\varphi = \varphi'_k$

Example:
For Projected Coordinate Reference System Tananarive (Paris) / Laborde Grid.

Parameters:

Ellipsoid:	International 1924		a = 6378388 metres	1/f = 297
		then	e = 0.081991890	e^2 = 0.006722670

Latitude of projection centre	φ_C	21 grads S	=	-0.329867229 rad
Longitude of projection centre	λ_C	49 grads E of Paris	=	51.5969213 grads E of Greenwich
			=	0.810482544 rad
Azimuth of initial line	α_C	21 grads	=	0.329867229 rad
Scale factor on initial line	k_C	0.9995		
Easting at projection centre	E_C	400000	metres	
Northings at projection centre	N_C	800000	metres	

Constants for the map projection:

B	=	1.002707541	φ_s	=	-0.328942879
R	=	6358218.319	C	=	-0.0002973474

Forward calculation for:

Latitude	φ	=	16°11'23.280"S		
			17.9886666667 grads S	=	-0.282565315 rad
Longitude	λ	=	44°27'27.260"E of Greenwich		
			46.800381173 grads E of Paris	=	0.735138668 rad

first gives :

L	=	-0.034645081	q	=	-0.285595283	P	=	-0.281790207
U	=	0.998343010	V	=	-0.046948995	W	=	-0.033271994
d	=	0.999446334	L'	=	-0.046992297	P'	=	-0.033278135
H	=	0.046992297 - 0.033284279i	G	=	0.017487082 + 0.051075588i			

Then	Easting	E	=	188333.848 metres
	Northing	N	=	1098841.091 metres

Reverse calculation for same easting and northing first gives:

G	=	0.017487082 + 0.051075588i	H_0	=	0.047000760 - 0.033290167i	H_1	=	0.999820949 - 0.000001503i
H_k	=	0.046992297 - 0.033284279i	L'	=	-0.046992297	P'	=	-0.033278136
U'	=	0.959982752	V'	=	-0.033271994	W'	=	-0.278075693
d	=	0.960559165	L	=	-0.3464508142	P	=	-0.281790207
q'	=	-0.284527565	φ'_0	=	-0.280764449			

Then	Latitude	φ	=	-0.282565315 rad	=	17.9886666667 grads S
					=	16°11'23.280"S
	Longitude	λ	=	0.735138668 rad	=	46.8003811733 grads East of Paris
					=	44°27'27.260"E of Greenwich

Comparing the Oblique Mercator method as an approximation of the full Laborde formula:

Latitude	Greenwich Longitude	Using Laborde formula Northing X	Using Laborde formula Easting Y	Using Oblique Mercator Northing X	Using Oblique Mercator Easting Y	dX (m)	dY (m)
18°54'S	47°30'E	799665.521	511921.054	799665.520	511921.054	0.00	0.00
16°12'S	44°24'E	1097651.447	182184.982	1097651.426	182184.985	0.02	0.00
25°40'S	45°18'E	50636.222	285294.334	50636.850	285294.788	0.63	0.45
12°00'S	49°12'E	1561109.146	701354.056	1561109.350	701352.935	0.20	1.12

1.3.7 Stereographic

The Stereographic projection may be imagined to be a projection of the earth's surface onto a plane in contact with the earth at a single tangent point from a projection point at the opposite end of the diameter through that tangent point.

This projection is best known in its polar form and is frequently used for mapping polar areas where it complements the Universal Transverse Mercator used for lower latitudes. Its spherical form has also been widely used by the US Geological Survey for planetary mapping and the mapping at small scale of continental hydrocarbon provinces. In its transverse or oblique ellipsoidal forms it is useful for mapping limited areas centred on the point where the plane of the projection is regarded as tangential to the ellipsoid., e.g. the Netherlands. The tangent point is the origin of the projected coordinate system and the meridian through it is regarded as the central meridian. In order to reduce the scale error at the extremities of the projection area it is usual to introduce a scale factor of less than unity at the origin such that a unitary scale factor applies on a near circle centred at the origin and some distance from it.

The coordinate conversion from geographical to projected coordinates is executed via the distance and azimuth of the point from the centre point or origin. For a sphere the formulas are relatively simple. For the ellipsoid the parameters defining the conformal sphere at the tangent point as origin are first derived. The conformal latitudes and longitudes are substituted for the geodetic latitudes and longitudes of the spherical formulas for the origin and the point.

An alternative approach is given by Snyder, where, instead of defining a single conformal sphere at the origin point, the conformal latitude at each point on the ellipsoid is computed. The conformal longitude is then always equivalent to the geodetic longitude. This approach is a valid alternative to that given here, but gives slightly different results away from the origin point. The USGS formula is therefore considered by OGP to be a different coordinate operation method to that described here.

1.3.7.1 Oblique and Equatorial Stereographic cases
(EPSG dataset coordinate operation method code 9809)

Given the geodetic origin of the projection at the tangent point (φ_O, λ_O), the parameters defining the conformal sphere are:

$$R = (\rho_O \ \nu_O)^{0.5}$$
$$n = \{1+[(e^2 \cos^4\varphi_O) / (1 - e^2)]\}^{0.5}$$
$$c = (n + \sin\varphi_O) (1 - \sin\chi_O) / [(n - \sin\varphi_O) (1 + \sin(\chi_O)]$$

where:
$$\sin \chi_O = (w_1 - 1) / (w_1 + 1)$$
$$w_1 = [S_1 (S_2)^e]^n$$
$$S_1 = (1 + \sin\varphi_O)/(1 - \sin\varphi_O)$$
$$S_2 = (1 - e \sin\varphi_O)/(1 + e \sin\varphi_O)$$

The conformal latitude and longitude of the origin (χ_O, Λ_O) are then computed from :

$$\chi_O = \sin^{-1} [(w_2 - 1)/(w_2 + 1)]$$

where S_1 and S_2 are as above and $w_2 = c\ [S_1\ (S_2)^{e}]^n = c\ w_1$

$$\Lambda_O = \lambda_O$$

For any point with geodetic coordinates (φ,λ) the equivalent conformal latitude and longitude (χ , Λ) are then computed from

$$\Lambda = n(\lambda - \Lambda_O) + \Lambda_O$$

and

$$\chi = \sin^{-1} [(w - 1)/(w + 1)]$$

where $w = c\ [S_a\ (S_b)^{e}]^n$
$S_a = (1 + \sin\varphi) / (1 - \sin\varphi)$
$S_b = (1 - e.\sin\varphi) / (1 + e.\sin\varphi)$

Then

$$E = FE + 2\ R\ k_O \cos\chi \sin(\Lambda - \Lambda_0) / B$$

and

$$N = FN + 2\ R\ k_O [\sin\chi \cos\chi_O - \cos\chi \sin\chi_O \cos(\Lambda - \Lambda_O)] / B$$

where $B = [1+\sin\chi \sin\chi_O + \cos\chi \cos\chi_O \cos(\Lambda - \Lambda_O)]$

The reverse formulas to compute the geodetic coordinates from the grid coordinates involves computing the conformal values, then the isometric latitude and finally the geodetic values.

The parameters of the conformal sphere and conformal latitude and longitude at the origin are computed as above. Then for any point with Stereographic grid coordinates (E,N) :

$$\chi = \chi_O + 2 \tan^{-1}\{[(N - FN) - (E - FE) \tan (j/2)] / (2\ Rk_O)\}$$

$$\Lambda = j + 2\ i + \Lambda_O$$

where $g = 2\ Rk_O \tan (\pi/4 - \chi_O/ 2)$
$h = 4\ Rk_O \tan \chi_O + g$
$i = \tan^{-1} \{(E - FE) / [h + (N - FN)]\}$
$j = \tan^{-1} \{(E - FE) / [g - (N - FN)]\} - i$

Geodetic longitude $\lambda = (\Lambda - \Lambda_O) / n + \Lambda_O$

Isometric latitude $\psi = 0.5 \ln \{(1 + \sin\chi) / [c (1 - \sin\chi)]\} / n$

First approximation $\varphi_1 = 2 \tan^{-1} \mathbf{e}^{\Psi} - \pi / 2$ where **e**=base of natural logarithms.

ψ_i = isometric latitude at φ_i

where $\psi_i= \ln\{[\tan(\varphi_i/2 +\pi/4)]\ [(1 - e \sin\varphi_i)/(1 + e \sin\varphi_i)]^{(e/2)}\}$

Then iterate $\varphi_{i+1} = \varphi_i - (\psi_i - \psi) \cos \varphi_i (1 - e^2 \sin^2\varphi_i) / (1 - e^2)$

until the change in φ is sufficiently small.

If the projection is the equatorial case, φ_O and χ_O will be zero degrees and the formulas are simplified as a result, but the above formulas remain valid.

For the polar version, φ_O and χ_O will be 90 degrees and the formulas become indeterminate. See below for formulas for the polar case.

For stereographic projections centred on points in the southern hemisphere, the signs of E, N, λ_O and λ must be reversed to be used in the equations and φ will be negative anyway as a southerly latitude.

Example:
For Projected Coordinate Reference System: Amersfoort / RD New

Parameters:

Ellipsoid:	Bessel 1841	a = 6377397.155 metres	1/f = 299.15281
	then	e = 0.08169683	

Latitude of natural origin	φ_O	52°09'22.178"N	=	0.910296727 rad
Longitude of natural origin	λ_O	5°23'15.500"E	=	0.094032038 rad
Scale factor at natural origin	k_O	0.9999079		
False easting	FE	155000.00	metres	
False northing	FN	463000.00	metres	

Forward calculation for:

Latitude	φ	=	53°N	=	0.925024504 rad
Longitude	λ	=	6°E	=	0.104719755 rad

first gives the conformal sphere constants:

ρ_O = 6374588.71 $\quad$ ν_O = 6390710.613
R = 6382644.571 $\quad$ n = 1.000475857 $\quad$ c = 1.007576465

where S_1 = 8.509582274 $\quad$ S_2 = 0.878790173 $\quad$ w_1 = 8.428769183
$\sin \chi_O$ = 0.787883237

w_2 = 8.492629457 $\quad$ χ_O = 0.909684757 $\quad$ $\Lambda_O = \lambda_O$ = 0.094032038 rad

For the point (φ,λ) $\quad$ χ = 0.924394997 $\quad$ Λ = 0.104724841 rad

hence B = 1.999870665
and $\quad$ E = 196105.283 m $\quad$ N = 557057.739 m

Reverse calculation for the same Easting and Northing (196105.28E, 557057.74N) first gives:

g = 4379954.188 $\quad$ h = 37197327.960 $\quad$ i = 0.001102255 $\quad$ j = 0.008488122

then Λ = 0.10472467 whence λ = 0.104719584 rad = 6°E

Also χ = 0.924394767 and ψ = 1.089495123
Then φ_1 = 0.921804948 $\quad$ ψ_1 = 1.084170164

$\varphi_2 = 0.925031162$ $\psi_2 = 1.089506925$
$\varphi_3 = 0.925024504$ $\psi_3 = 1.089495505$
$\varphi_4 = 0.925024504$

Then Latitude φ = 53°00'00.000"N
Longitude λ = 6°00'00.000"E

1.3.7.2 Polar Stereographic

For the polar sterographic projection, three variants are recognised, differentiated by their defining parameters. In the basic variant (**variant A**) the latitude of origin is either the north or the south pole, at which is defined a scale factor at the natural origin, the meridian along which the northing axis increments and along which intersecting parallels increment *towards the north pole* (the longitude of origin), and false grid coordinates. In **variant B** instead of the scale factor at the pole being defined, the (non-polar) latitude at which the scale is unity – the standard parallel – is defined. In **variant C** the latitude of a standard parallel along which the scale is unity is defined; the intersection of this parallel with the longitude of origin is the false origin, at which grid coordinate values are defined.

	Method		
Parameter	Variant A	Variant B	Variant C
Latitude of natural origin (φ_O)	(note 1)	(note 2)	(note 2)
Latitude of standard parallel (φ_F)		x	x
Longitude of origin (λ_O)	x	x	x
Scale at natural origin (k_O)	x		
False easting (easting at natural origin = pole) (FE)	x	x	
False northing (northing at natural origin = pole) (FN)	x	x	
Easting at false origin (E_F)			x
Northing at false origin (N_F)			x

In all three variants the formulae for the **south pole case** are straightforward but some require modification for the **north pole case** to allow the longitude of origin going towards (as opposed to away from) the natural origin and for the anticlockwise increase in longitude value when viewed from the north pole (see figure 8). Several equations are common between the variants and cases.

Notes:
1. In variant A the parameter *Latitude of natural origin* is used only to identify which hemisphere case is required. The only valid entries are ±90° or equivalent in alternative angle units.
2. For variants B and C, whilst it is mathematically possible for the standard parallel to be in the opposite hemisphere to the pole at which is the projection natural origin, such an arrangement would be unsatisfactory from a cartographic perspective as the rate of change of scale would be excessive in the area of interest. The EPSG dataset therefore excludes the hemisphere of pole as a defining parameter for these variants. In the formulas that follow for these variants B and C, the hemisphere of pole is taken to be that of the hemisphere of the standard parallel.

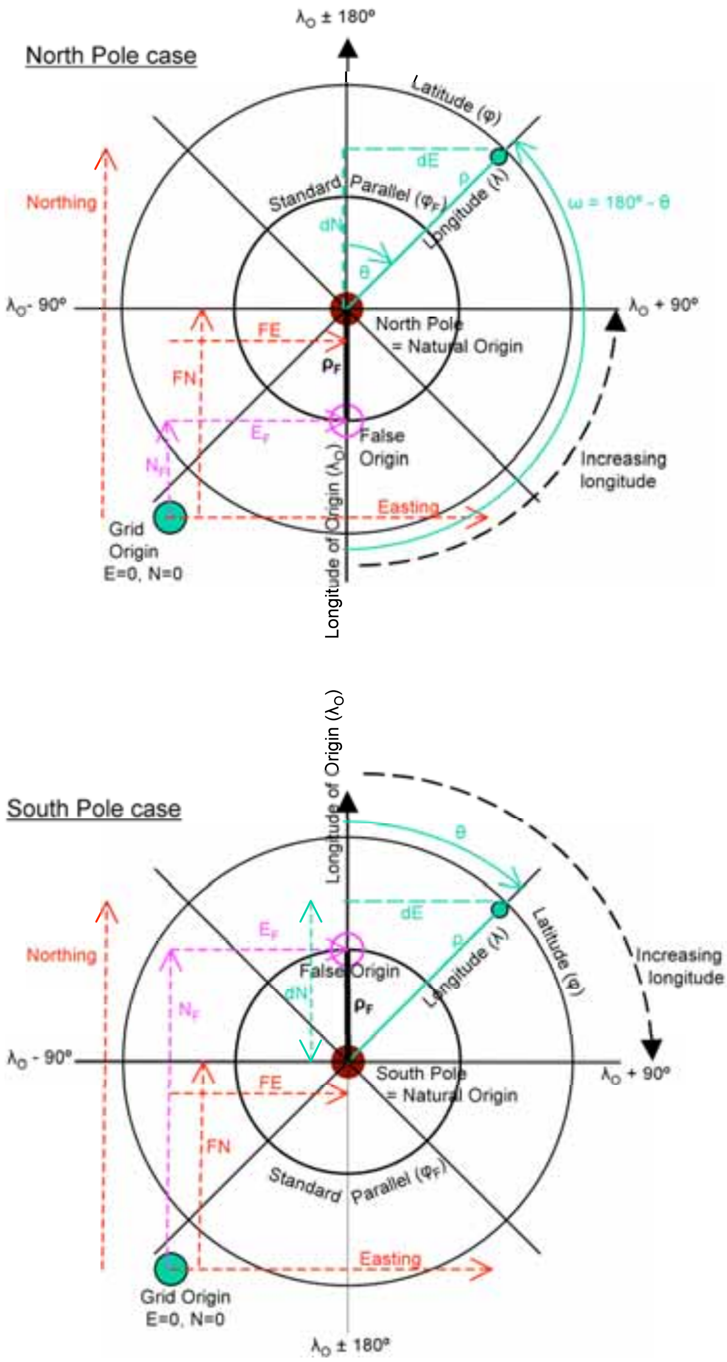

Figure 8. Key Diagram for Stereographic Projection

Polar Stereographic (Variant A) (EPSG dataset coordinate operation method code 9810).

For the forward conversion from latitude and longitude, for the south pole case

$$dE = \rho \sin(\theta)$$

and

$$dN = \rho \cos(\theta)$$

where $\theta = (\lambda - \lambda_O)$

Then

$$E = dE + FE = FE + \rho \sin(\lambda - \lambda_O)$$

$$N = dN + FN = FN + \rho \cos(\lambda - \lambda_O)$$

where

$t = \tan(\pi/4 + \varphi/2) / \{[(1 + e \sin\varphi) / (1 - e \sin\varphi)]^{(e/2)}\}$

$\rho = 2\, a\, k_O\, t / \{[(1+e)^{(1+e)} (1-e)^{(1-e)}]^{0.5}\}$

For the north pole case,

$dE = \rho \sin(\theta) = \rho \sin(\omega)$

$dN = \rho \cos(\theta) = -\rho \cos(\omega)$

where, as shown in figure 8, ω = longitude λ measured anticlockwise in the projection plane.

ρ and E are found as for the south pole case but

$t = \tan(\pi/4 - \varphi/2) \{[(1 + e \sin\varphi) / (1 - e \sin\varphi)]^{(e/2)}\}$

$N = FN - \rho \cos(\lambda - \lambda_O)$

For the reverse conversion from easting and northing to latitude and longitude,

$$\varphi = \chi + (e^2/2 + 5e^4/24 + e^6/12 + 13e^8/360) \sin(2\chi) + (7e^4/48 + 29e^6/240 + 811e^8/11520) \sin(4\chi) + (7e^6/120 + 81e^8/1120) \sin(6\chi) + (4279e^8/161280) \sin(8\chi)$$

where $\rho' = [(E - FE)^2 + (N - FN)^2]^{0.5}$

$t' = \rho' \{[(1+e)^{(1+e)} (1-e)^{(1-e)}]^{0.5}\} / 2\, a\, k_O$

and for the south pole case

$\chi = 2 \operatorname{atan}(t') - \pi/2$

but for the north pole case

$\chi = \pi/2 - 2 \operatorname{atan} t'$

Then for both north and south cases if E = FE, $\lambda = \lambda_O$

else for the south pole case

$\lambda = \lambda_O + \operatorname{atan}[(E - FE) / (N - FN)]$

and for the north pole case

$\lambda = \lambda_O + \operatorname{atan}[(E - FE) / -(N - FN)] = \lambda_O + \operatorname{atan}[(E - FE) / (FN - N)]$

Example:

For Projected Coordinate Reference System: WGS 84 / UPS North

Parameters:

Ellipsoid:	WGS 84	a = 6378137.0 metres	1/f = 298.2572236
	then	e = 0.081819191	

Latitude of natural origin	φ_O	90°00'00.000"N	=	1.570796327 rad
Longitude of natural origin	λ_O	0°00'00.000"E	=	0.0 rad
Scale factor at natural origin	k_O	0.994		
False easting	FE	2000000.00	metres	
False northing	FN	2000000.00	metres	

Forward calculation for:

Latitude	φ	=	73°N	=	1.274090354 rad
Longitude	λ	=	44°E	=	0.767944871 rad

t = 0.150412808

ρ = 1900814.564

whence

E = 3320416.75 m

N = 632668.43 m

Reverse calculation for the same Easting and Northing (3320416.75 E, 632668.43 N) first gives:

$\rho' = 1900814.566$
$t' = 0.150412808$
$\chi = 1.2722090$

Then	Latitude	φ	=	73°00'00.000"N
	Longitude	λ	=	44°00'00.000"E

Polar Stereographic (Variant B) (EPSG dataset coordinate operation method code 9829).

For the forward conversion from latitude and longitude:
for the south pole case

$$t_F = \tan(\pi/4 + \varphi_F/2) / \{[(1 + e \sin\varphi_F) / (1 - e \sin\varphi_F)]^{(e/2)}\}$$
$$m_F = \cos\varphi_F / (1 - e^2 \sin^2\varphi_F)^{0.5}$$
$$k_O = m_F \{[(1+e)^{(1+e)} (1-e)^{(1-e)}]^{0.5}\} / (2*t_F)$$

then t, ρ, E and N are found as in the south pole case of variant A.

For the north pole case, m_F and k_O are found as for the south pole case above, but

$$t_F = \tan(\pi/4 - \varphi_F/2) \{[(1 + e \sin\varphi_F) / (1 - e \sin\varphi_F)]^{(e/2)}\}$$

Then t, ρ, E and N are found as in variant A.

For the reverse conversion from easting and northing to latitude and longitude, first k_O is found from m_F and t_F as in the forward conversion above, then φ and λ are found as for variant A.

Example:
For Projected Coordinate Reference System: WGS 84 / Australian Antarctic Polar Stereographic

Parameters:

Ellipsoid:	WGS 84	a = 6378137.0 metres	1/f = 298.2572236
	then	e = 0.081819191	

Latitude of standard parallel	φ_F	71°00'00.000"S	=	-1.239183769 rad
Longitude of origin	λ_O	70°00'00.000"E	=	1.221730476 rad
False easting	FE	6000000.00	metres	
False northing	FN	6000000.00	metres	

Forward calculation for:

Latitude	φ	=	75°00'00.000"S	=	-1.308996939 rad
Longitude	λ	=	120°00'00.000"E	=	2.094395102 rad

$t_F = 0.168407325$
$m_F = 0.326546781$
$k_O = 0.97276901$
$t = 0.132508348$
$\rho = 1638783.238$

whence

E = 7255380.79 m
N = 7053389.56 m

Reverse calculation for the same Easting and Northing (7255380.79 E, 7053389.56 N) first gives:

$t_F = 0.168407325$ $m_F = 0.326546781$ and $k_O = 0.97276901$

then $\rho' = 1638783.236$ $t' = 0.132508347$ $\chi = -1.3073146$

Then	Latitude	φ	=	75°00'00.000"S
	Longitude	λ	=	120°00'00.000"E

Polar Stereographic (Variant C) (EPSG dataset coordinate operation method code 9830).

For the forward conversion from latitude and longitude, for the south pole case

$E = E_F + \rho \sin(\lambda - \lambda_O)$

$N = N_F - \rho_F + \rho \cos(\lambda - \lambda_O)$

where

m_F is found as in variant B = $\cos\varphi_F / (1 - e^2 \sin^2\varphi_F)^{0.5}$

t_F is found as in variant B = $\tan(\pi/4 + \varphi_F/2) / \{[(1 + e \sin\varphi_F) / (1 - e \sin\varphi_F)]^{(e/2)}\}$

t is found as in variants A and B = $\tan(\pi/4 + \varphi/2) / \{[(1 + e \sin\varphi) / (1 - e \sin\varphi)]^{(e/2)}\}$

$\rho_F = a\, m_F$

$\rho = \rho_F\, t / t_F$

For the north pole case, m_F, ρ_F, ρ and E are found as for the south pole case but

t_F is found as in variant B = $\tan(\pi/4 - \varphi_F/2)\, \{[(1 + e \sin\varphi_F) / (1 - e \sin\varphi_F)]^{(e/2)}\}$

t is found as in variants A and B = $\tan(\pi/4 - \varphi/2)\, \{[(1 + e \sin\varphi) / (1 - e \sin\varphi)]^{(e/2)}\}$

$N = N_F + \rho_F - \rho \cos(\lambda - \lambda_O)$

For the reverse conversion from easting and northing to latitude and longitude,

$$\varphi = \chi + (e^2/2 + 5e^4/24 + e^6/12 + 13e^8/360) \sin(2\chi) + (7e^4/48 + 29e^6/240 + 811e^8/11520) \sin(4\chi) + (7e^6/120 + 81e^8/1120) \sin(6\chi) + (4279e^8/161280) \sin(8\chi)$$

(as for variants A and B)

where for the south pole case

$\rho' = [(E-E_F)^2 + (N - N_F + \rho_F)^2]^{0.5}$

$t' = \rho'\, t_F / \rho_F$

$\chi = 2\, \mathrm{atan}(t') - \pi/2$

and where m_F and t_F are as for the forward conversion

For the reverse conversion north pole case, m_F, t_F and ρ_F are found as for the north pole case of the forward conversion, and

$\rho' = [(E-E_F)^2 + (N - N_F - \rho_F)^2]^{0.5}$

t' is found as for the south pole case of the reverse conversion = $\rho'\, t_F / \rho_F$

$\chi = \pi/2 - 2\, \mathrm{atan}\, t'$

Then for for both north and south pole cases

if $E = E_F$, $\lambda = \lambda_O$

else for the south pole case

$\lambda = \lambda_O + \mathrm{atan}\, [(E - E_F) / (N - N_F + \rho_F)]$

and for the north pole case

$\lambda = \lambda_O + \mathrm{atan}\, [(E - E_F) / -(N - N_F - \rho_F)] = \lambda_O + \mathrm{atan}\, [(E - E_F) / (N_F + \rho_F - N)]$

Example:
For Projected Coordinate Reference System: Petrels 1972 / Terre Adelie Polar Stereographic

Parameters:

Ellipsoid:	International 1924	a = 6378388.0 metres	1/f = 297.0
	then	e = 0.081991890	

Latitude of false origin	φ_F	67°00'00.000"S	=	-1.169370599 rad
Longitude of origin	λ_O	140°00'00.000"E	=	2.443460953 rad
Easting at false origin	E_F	300000.00	metres	
Northing at false origin	N_F	200000.00	metres	

Forward calculation for:

Latitude	φ	=	66°36'18.820"S	=	-1.162480524 rad
Longitude	λ	=	140°04'17.040"E	=	2.444707118 rad

m_F = 0.391848769
ρ_F = 2499363.488
t_F = 0.204717630
t = 0.208326304
ρ = 2543421.183

whence
E = 303169.52 m
N = 244055.72 m

Reverse calculation for the same Easting and Northing (303169.522 E, 244055.721 N) first gives:
ρ' = 2543421.183
t' = 0.208326304
χ = -1.1600190

Then	Latitude	φ	=	66°36'18.820"S
	Longitude	λ	=	140°04'17.040"E

1.3.8 New Zealand Map Grid
(EPSG dataset coordinate operation method code 9811)

This projection system typifies the recent development in the design and formulation of map projections where, by more complex mathematics yielding formulas readily handled by modern computers, it is possible to maintain the conformal property and minimise scale distortion over the total extent of a country area regardless of shape. Thus both North and South Islands of New Zealand, previously treated not very satisfactorily in two zones of a Transverse Mercator projection, can now be projected as one zone of what resembles most closely a curved version Oblique Mercator but which, instead of being based on a minimum scale factor straight central line, has a central line which is a complex curve roughly following the trend of both North and South Islands. The projected coordinate reference system achieves this by a form of double projection where a conformal projection of the ellipsoid is first made to say an oblique Stereographic projection and then the Cauchy-Riemann equations are invoked in order to further project the rectangular coordinates on this to a modification in which lines of constant scale can be made to follow other than the normal great or small circles of Central meridians or standard parallels. The mathematical treatment of the New Zealand Map Grid is covered by a publication by New Zealand Department of Lands and Survey Technical Circular 1973/32 by I.F.Stirling.

1.3.9 Tunisia Mining Grid

(EPSG dataset coordinate operation method code 9816)

This grid is used as the basis for mineral leasing in Tunisia. Lease areas are approximately 2 x 2 km or 400 hectares. The corners of these blocks are defined through a six figure grid reference where the first three digits are an easting in kilometres and the last three digits are a northing. The latitudes and longitudes for block corners at 2 km intervals are tabulated in a mining decree dated 1st January 1953. From this tabulation in which geographical coordinates are given to 5 decimal places it can be seen that:

a) the minimum easting is 94 km, on which the longitude is 5.68989 grads east of Paris.
b) the maximum easting is 490 km, on which the longitude is 10.51515 grads east of Paris.
c) each 2 km grid easting interval equals 0.02437 grads.
d) the minimum northing is 40 km, on which the latitude is 33.39 grads.
e) the maximum northing is 860 km, on which the latitude is 41.6039 grads.
f) between 40 km N and 360 km N, each 2 km grid northing interval equals 0.02004 grads.
g) between 360 km N and 860 km N, each 2 km grid northing interval equals 0.02003 grads.

This grid could be considered to be two equidistant cylindrical projection zones, north and south of the 360 km northing line. However this would require the introduction of two spheres of unique dimensions. OGP has therefore implemented the Tunisia mining grid as a coordinate conversion method in its own right. Formulas are:

Grads from Paris

φ (grads) = 36.5964 + [(N – 360) * A]
where N is in kilometres and A = 0.010015 if N > 360, else A = 0.01002.

λ_{Paris} (grads) = 7.83445 + [(E – 270) * 0.012185], where E is in kilometres.

The reverse formulas are:

E (km) = 270 + [(λ_{Paris} – 7.83445) / 0.012185] where λ_{Paris} is in grads.

N (km) = 360 + [(φ – 36.5964) / B]
where φ is in grads and B = 0.010015 if φ > 36.5964, else B = 0.01002.

Degrees from Greenwich

Modern practice in Tunisia is to quote latitude and longitude in degrees with longitudes referenced to the Greenwich meridian. The formulas required in addition to the above are:

φ_d (degrees) = (φ_g * 0.9) where φ_g is in grads.
$\lambda_{Greenwich}$ (degrees) = [(λ_{Paris} + 2.5969213) * 0.9] where λ_{Paris} is in grads.

φ_g (grads) = (φ_d / 0.9) where φ_d is in decimal degrees.
λ_{Paris} (grads) = [($\lambda_{Greenwich}$ / 0.9) – 2.5969213)] where $\lambda_{Greenwich}$ is in decimal degrees.

Example:

For grid location 302598,
Latitude φ = 36.5964 + [(598 – 360) * A]. As N > 360, A = 0.010015.
φ = 38.97997 grads = 35.08197 degrees.

Longitude λ = 7.83445 + [(E – 270) * 0.012185], where E = 302.
λ = 8.22437 grads east of Paris = 9.73916 degrees east of Greenwich.

1.3.10 American Polyconic

(EPSG dataset coordinate operation method code 9818)

This projection was popular before the advent of modern computers due to its ease of mechanical construction, particularly in the United States. It is neither conformal nor equal area, and is distortion-free only along the longitude of origin. A modified form of the polyconic projection was adopted in 1909 for the International Map of the World series of 1/1,000,000 scale topographic maps. A general study of the polyconic family of projections by Oscar Adams of the US Geological Survey was published in 1919 (and reprinted in 1934).

The formulas to derive the projected Easting and Northing coordinates are:

If $\varphi = 0$:

Easting, $E = FE + a(\lambda - \lambda_O)$
Northing, $N = FN - M_O$

If φ is not zero:

Easting, $E = FE + \nu \cot\varphi \sin L$

Northing, $N = FN + M - Mo + \nu \cot\varphi(1 - \cos L)$

where $L = (\lambda - \lambda_O)\sin\varphi$
$\nu = a / (1 - e^2\sin^2\varphi)^{0.5}$
$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024 + ...)\sin 2\varphi$
$+ (15e^4/256 + 45e^6/1024 + ...)\sin 4\varphi - (35e^6/3072 + ...)\sin 6\varphi + ...]$
with φ in radians and M_O for φ_O, the latitude of the origin, derived in the same way.

The reverse formulas to convert Easting and Northing projected coordinates to latitude and longitude require iteration. This iteration will not converge if $(\lambda - \lambda_O) > 90°$ but the projection should not be used in that range.

First M_O is calculated using the formula for M given in the forward case. Then:

If $M_O = (N-FN)$ then:

$\varphi = 0$
$\lambda = \lambda_O + (E-FE)/a$

If M_O does not equal (N-FN) then:

$A = [M_O + (N-FN)] / a$
$B = A^2 + \{[(E-FE)^2] / a^2\}$

C =
M is found
H = Mn
J = H / a
$\varphi'' = \varphi' - [A(C J + 1) - J - 0.5C(J^2 + B)] /$
$\{e^2 \sin 2\varphi' (J^2 + B - 2A J) / 4C + (A - J)(C H - [2 / \sin 2\varphi']) - H\}$

Then after solution of φ

$$\lambda = \lambda_O + \{\text{asin}[(E\text{-}FE)\ C / a]\} / \sin\varphi$$

1.3.11 Lambert Azimuthal Equal Area

(EPSG dataset coordinate operation method code 9820)

Oblique aspect

To derive the projected coordinates of a point, geodetic latitude (φ) is converted to authalic latitude (ß). The formulae to convert geodetic latitude and longitude (φ, λ) to Easting and Northing are:

$$\text{Easting, } E = FE + \{(B * D)\ [\cos ß\ \sin(\lambda - \lambda_O)]\}$$

$$\text{Northing, } N = FN + (B / D)\ \{(\cos ß_O\ \sin ß) - [\sin ß_O\ \cos ß\ \cos(\lambda - \lambda_O)]\}$$

where

$$B = R_q\ (2 / \{1 + \sin ß_O\ \sin ß + [\cos ß_O\ \cos ß\ \cos(\lambda - \lambda_O)]\})^{0.5}$$

$$D = a\ [\cos \varphi_O / (1 - e^2 \sin^2 \varphi_O)^{0.5}] / (R_q\ \cos ß_O)$$

$$R_q = a\ (q_P / 2)^{0.5}$$

$$ß = \text{asin}\ (q / q_P)$$

$$ß_O = \text{asin}\ (q_O / q_P)$$

$$q = (1 - e^2)\ ([\sin \varphi / (1 - e^2\sin^2\varphi)] - \{[1/(2e)]\ \ln [(1 - e \sin \varphi) / (1 + e \sin \varphi)]\})$$

$$q_O = (1 - e^2)\ ([\sin \varphi_O / (1 - e^2\sin^2\varphi_O)] - \{[1/(2e)]\ \ln [(1 - e \sin \varphi_O) / (1 + e \sin \varphi_O)]\})$$

$$q_P = (1 - e^2)\ ([\sin \varphi_P / (1 - e^2\sin^2\varphi_P)] - \{[1/(2e)]\ \ln [(1 - e \sin \varphi_P) / (1 + e \sin \varphi_P)]\})$$

where $\varphi_P = \pi/2$ radians, thus

$$q_P = (1 - e^2)\ ([1 / (1 - e^2)] - \{[1/(2e)]\ \ln [(1 - e) / (1 + e)]\})$$

The reverse formulas to derive the geodetic latitude and longitude of a point from its Easting and Northing values are:

$$\varphi = ß' + [(e^2/3 + 31e^4/180 + 517e^6/5040)\ \sin 2ß'] + [(23e^4/360 + 251e^6/3780)\ \sin 4ß'] + [(761e^6/45360)\ \sin 6ß']$$

$$\lambda = \lambda_O + \text{atan}\ \{(E\text{–}FE)\ \sin C / [D\ \rho\ \cos ß_O\ \cos C - D^2\ (N\text{–}FN)\ \sin ß_O\ \sin C]\}$$

where

$$ß' = \text{asin}\{(\cos C\ \sin ß_O) + [(D\ (N\text{–}FN)\ \sin C\ \cos ß_O) / \rho]\}$$

$$C = 2\ \text{asin}(\rho / 2\ R_q)$$

$$\rho = \{[(E\text{–}FE)/D]^2 + [D\ (N - FN)]^2\}^{0.5}$$

and D, R_q, and $ß_O$ are as in the forward equations.

Example

For Projected Coordinate Reference System: ETRS89 / ETRS-LAEA

Parameters:

Ellipsoid:	GRS 1980	a = 6378137.0 metres	1/f = 298.2572221
	then	e = 0.081819191	

Latitude of natural origin	φ_O	52°00'00.000"N	=	0.907571211 rad
Longitude of natural origin	λ_O	10°00'00.000"E	=	0.174532925 rad
False easting	FE	4321000.00	metres	
False northing	FN	3210000.00	metres	

Forward calculation for:

Latitude	φ	=	50°00'00.000"N	=	0.872664626 rad
Longitude	λ	=	5°00'00.000"E	=	0.087266463 rad

First gives

q_P =	1.995531087	q_O =	1.569825704
q =	1.525832247	R_q =	6371007.181
$ß_O$ =	0.905397517	$ß$ =	0.870458708
D =	1.000425395	B =	6374393.455

whence

E = 3962799.45 m
N = 2999718.85 m

Reverse calculation for the same Easting and Northing (3962799.45 E, 2999718.85 N) first gives:

ρ = 415276.208
C = 0.065193736
$ß'$ = 0.870458708

Then Latitude φ = 50°00'00.000"N
Longitude λ = 5°00'00.000"E

Polar aspect

For the polar aspect of the Lambert Azimuthal Equal Area projection, some of the above equations are indeterminate. Instead, for the forward case from latitude and longitude (φ, λ) to Easting (E) and Northing (N):

For the north polar case:

Easting, E = FE + [ρ sin($\lambda - \lambda_O$)]
Northing, N = FN – [ρ cos($\lambda - \lambda_O$)]

where

$\rho = a (q_P - q)^{0.5}$

and q_P and q are found as for the general case above.

For the south polar case:

Easting, E = FE + [ρ sin($\lambda - \lambda_O$)]
Northing, N = FN + [ρ cos($\lambda - \lambda_O$)]

where

$\rho = a (q_P + q)^{0.5}$

and q_P and q are found as for the general case above.

For the reverse formulas to derive the geodetic latitude and longitude of a point from its Easting and Northing:

$\varphi = ß' + [(e^2/3 + 31e^4/180 + 517e^6/5040) \sin 2ß'] + [(23e^4/360 + 251e^6/3780) \sin 4ß'] + [(761e^6/45360) \sin 6ß']$

as for the oblique case, but where

$ß' = \pm\text{asin}\, [1- \rho^2 / (a^2\{1- [(1- e^2)/2e] \ln[(1-e)/(1+e)]\})]$, taking the sign of φ_O

and $\rho = \{[(E - FE)]^2 + [(N - FN)]^2\}^{0.5}$

Then

$\lambda = \lambda_O + \text{atan}\, [(E - FE) / (N - FN)]$ for the south pole case

and

$\lambda = \lambda_O + \text{atan}\, [(E - FE) / -(N - FN)] = \lambda_O + \text{atan}\, [(E - FE) / (FN - N)]$ for the north pole case.

1.3.11.1 Lambert Azimuthal Equal Area (Spherical)
(EPSG dataset coordinate operation method code 1027)

The US National Atlas uses the spherical form of the oblique case, so exceptionally OGP includes this method in the EPSG dataset. See USGS Professional Paper 1395, "Map Projections - A Working Manual" by John P. Snyder for formulas and example.

R is the radius of the sphere and will normally be one of the CRS parameters. If the figure of the earth used is an ellipsoid rather than a sphere then R should be calculated as the radius of the authalic sphere using the formula for R_A given in section 1.2 of this Guidance Note, table 3. Note however that if applying spherical formula to ellipsoidal coordinates, the authalic projection properties are not preserved.

1.3.12 Lambert Cylindrical Equal Area
(EPSG dataset coordinate operation method code 9835)

See USGS Professional Paper 1395, "Map Projections - A Working Manual" by John P. Snyder for formulas and example.

1.3.12.1 Lambert Cylindrical Equal Area (Spherical)
(EPSG dataset coordinate operation method code 9834)

For the forward calculation for the normal aspect of the projection in which φ_1 is the latitude of the standard parallel:

$$E = FE + R\,(\lambda - \lambda_O)\cos(\varphi_1)$$
$$N = FN + R\sin(\varphi) / \cos(\varphi_1)$$

where φ_1, φ and λ are expressed in radians

R is the radius of the sphere and will normally be one of the CRS parameters. If the figure of the earth used is an ellipsoid rather than a sphere then R should be calculated as the radius of the authalic sphere using the formula for R_A given in section 1.2, table 3.

For the reverse calculation:

$$\varphi = \text{asin}\{[(N - FN) / R]\cos(\varphi_1)\}$$
$$\lambda = \lambda_O + \{[E - FE] / [R\cos(\varphi_1)]\}$$

where R is as for the forward method.

See USGS Professional Paper 1395, "Map Projections - A Working Manual" by John P. Snyder for formulas for oblique and polar aspects and examples.

1.3.13 Albers Equal Area
(EPSG dataset coordinate operation method code 9822)

To derive the projected coordinates of a point, geodetic latitude (φ) is converted to authalic latitude (ß). The formulas to convert geodetic latitude and longitude (φ, λ) to Easting (E) and Northing (N) are:

$$\text{Easting (E)} = E_F + (\rho \sin\theta)$$
$$\text{Northing (N)} = N_F + \rho_O - (\rho \cos\theta)$$

where

$\theta = n\ (\lambda - \lambda_O)$
$\rho = [a\ (C - n\ \alpha)^{0.5}] / n$
$\rho_O = [a\ (C - n\ \alpha_O)^{0.5}] / n$

and

$C = m_1^2 + (n\ \alpha_1)$
$n = (m_1^2 - m_2^2) / (\alpha_2 - \alpha_1)$
$m_1 = \cos\varphi_1 / (1 - e^2\sin^2\varphi_1)^{0.5}$
$m_2 = \cos\varphi_2 / (1 - e^2\sin^2\varphi_2)^{0.5}$
$\alpha = (1 - e^2) \{[\sin\varphi / (1 - e^2\sin^2\varphi)] - [1/(2e)] \ln [(1 - e \sin\varphi) / (1 + e \sin\varphi)]\}$
$\alpha_O = (1 - e^2) \{[\sin\varphi_O / (1 - e^2\sin^2\varphi_O)] - [1/(2e)] \ln [(1 - e \sin\varphi_O) / (1 + e \sin\varphi_O)]\}$
$\alpha_1 = (1 - e^2) \{[\sin\varphi_1 / (1 - e^2\sin^2\varphi_1)] - [1/(2e)] \ln [(1 - e \sin\varphi_1) / (1 + e \sin\varphi_1)]\}$
$\alpha_2 = (1 - e^2) \{[\sin\varphi_2 / (1 - e^2\sin^2\varphi_2)] - [1/(2e)] \ln [(1 - e \sin\varphi_2) / (1 + e \sin\varphi_2)]\}$

The reverse formulas to derive the geodetic latitude and longitude of a point from its Easting and Northing values are:

$$\varphi = \beta' + (e^2/3 + 31e^4/180 + 517e^6/5040)\ \sin 2\beta'] + [(23e^4/360 + 251e^6/3780)\ \sin 4\beta'] + [(761e^6/45360)\ \sin 6\beta']$$

$$\lambda = \lambda_O + (\theta / n)$$

where

$\beta' = \operatorname{asin}(\alpha' / \{1 - [(1 - e^2) / (2\ e)]\ \ln [(1 - e) / (1 + e)]$
$\alpha' = [C - (\rho^2\ n^2 / a^2)] / n$
$\rho = \{(E - E_F)^2 + [\rho_O - (N - N_F)]^2\}^{0.5}$
$\theta = \operatorname{atan} [(E - E_F) / [\rho_O - (N - N_F)]$

and C, n and ρ_O are as in the forward equations.

Example
See USGS Professional Paper 1395, "Map Projections - A Working Manual" by John P. Snyder.

1.3.14 <u>Equidistant Cylindrical</u>

(EPSG dataset coordinate operation method code 1028)

The characteristics of the Equidistant Cylindrical projection are that the scale is true along two standard parallels equidistant from the equator (or at the equator if that is the standard parallel) and along the meridians. The formulas usually given for this method employ spherical equations with a mean radius of curvature sphere calculated from the latitude of standard parallel. This is a compromise, often satisfactory for the low resolution purposes to which it is put. However in the spherical implementation the distance is not true along the meridians nor along the standard parallel(s). Spherical formulas are given in section 1.13.14.1 below.

The ellipsoidal forward equations to convert latitude and longitude to easting and northing are

$E = FE + \nu_1 \cos\varphi_1 (\lambda - \lambda_O)$
$N = FN + M$

where

$\nu_1 = a /(1 - e^2\sin^2\varphi_1)^{0.5}$ (see section 1.2 table 3)

and

$$M = a(1 - e^2)\int_0^{\varphi} (1 - e^2 \sin^2 \varphi)^{-3/2}\, d\varphi \tag{1}$$

or

$$M = a[\int_0^{\varphi} (1 - e^2 \sin^2 \varphi)^{1/2} d\varphi - e^2 \sin\varphi \cos\varphi / (1 - e^2 \sin^2 \varphi)^{1/2}] \qquad (2)$$

The first calculation (1) of M above contains an elliptic integral of the third kind. The alternative calculation (2) of M contains an elliptic integral of the second kind. If software supports the functions for these integrals, then the functions can be used directly. Otherwise, the value of M can be computed through a series equation. The following series equation is adequate for any ellipsoid with a flattening of 1/290 or less, which covers all earth-based ellipsoids of record.

$$\begin{aligned}
M = a[&(1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4 - \tfrac{5}{256}e^6 - \tfrac{175}{16384}e^8 - \tfrac{441}{65536}e^{10} - \tfrac{4851}{1048576}e^{12} - \tfrac{14157}{4194304}e^{14})\varphi \\
&+ (-\tfrac{3}{8}e^2 - \tfrac{3}{32}e^4 - \tfrac{45}{1024}e^6 - \tfrac{105}{4096}e^8 - \tfrac{2205}{131072}e^{10} - \tfrac{6237}{524288}e^{12} - \tfrac{297297}{33554432}e^{14})\sin 2\varphi \\
&+ (\tfrac{15}{256}e^4 + \tfrac{45}{1024}e^6 + \tfrac{525}{16384}e^8 + \tfrac{1575}{65536}e^{10} + \tfrac{155925}{8388608}e^{12} + \tfrac{495495}{33554432}e^{14})\sin 4\varphi \\
&+ (-\tfrac{35}{3072}e^6 - \tfrac{175}{12288}e^8 - \tfrac{3675}{262144}e^{10} - \tfrac{13475}{1048576}e^{12} - \tfrac{385385}{33554432}e^{14})\sin 6\varphi \\
&+ (\tfrac{315}{131072}e^8 + \tfrac{2205}{524288}e^{10} + \tfrac{43659}{8388608}e^{12} + \tfrac{189189}{33554432}e^{14})\sin 8\varphi \\
&+ (-\tfrac{693}{1310720}e^{10} - \tfrac{6237}{5242880}e^{12} - \tfrac{297297}{167772160}e^{14})\sin 10\varphi \\
&+ (\tfrac{1001}{8388608}e^{12} + \tfrac{11011}{33554432}e^{14})\sin 12\varphi \\
&+ (-\tfrac{6435}{234881024}e^{14})\sin 14\varphi]
\end{aligned}$$

The inverse equations are

$$\begin{aligned}
\lambda &= \lambda_0 + X/(\nu_1 \cos\varphi_1) = \lambda_0 + X(1 - e^2 \sin^2 \varphi_1)^{1/2} / (a \cos\varphi_1) \\
\varphi &= \mu + (\tfrac{3}{2}n - \tfrac{27}{32}n^3 + \tfrac{269}{512}n^5 - \tfrac{6607}{24576}n^7)\sin 2\mu \\
&+ (\tfrac{21}{16}n^2 - \tfrac{55}{32}n^4 + \tfrac{6759}{4096}n^6)\sin 4\mu \\
&+ (\tfrac{151}{96}n^3 - \tfrac{417}{128}n^5 + \tfrac{87963}{20480}n^7)\sin 6\mu \\
&+ (\tfrac{1097}{512}n^4 - \tfrac{15543}{2560}n^6)\sin 8\mu \\
&+ (\tfrac{8011}{2560}n^5 - \tfrac{69119}{6144}n^7)\sin 10\mu \\
&+ (\tfrac{293393}{61440}n^6)\sin 12\mu \\
&+ (\tfrac{6845701}{860160}n^7)\sin 14\mu
\end{aligned}$$

where

X = E - FE
Y = N - FN
$$\mu = Y/[a(1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4 - \tfrac{5}{256}e^6 - \tfrac{175}{16384}e^8 - \tfrac{441}{65536}e^{10} - \tfrac{4851}{1048576}e^{12} - \tfrac{14157}{4194304}e^{14})]$$
$$n = \frac{1 - (1 - e^2)^{1/2}}{1 + (1 - e^2)^{1/2}}$$

Example
For Projected Coordinate Reference System: WGS84 / World Equidistant Cylindrical
Parameters:

Ellipsoid:	WGS 1984	a = 6378137.0 metres	1/f =298.257223563
	then	e = 0.08181919084262	

Latitude of first standard parallel	φ_1	=	0°00'00.000"N	=	0.0 rad
Longitude of natural origin	λ_O	=	0°00'00.000"E	=	0.0 rad
False easting	FE	=	0.00	metres	
False northing	FN	=	0.00	metres	

Forward calculation for:

Latitude	φ	=	55°00'00.000"N	=	0.959931086 rad
Longitude	λ	=	10°00'00.000"E	=	0.174532925 rad

First gives

Radius of curvature in prime vertical at φ_1	ν_1 =	6378137.0
Radius of curvature of parallel at φ_1	$\nu_1 \cos \varphi_1$ =	6378137.0
Meridional arc distance from equator to φ	M =	6097230.3131

whence
E = 1113194.91 m
N = 6097230.31 m

Reverse calculation for the same Easting and Northing (1113194.91 E, 6097230.31 N) first gives:

Rectifying latitude (radians)	μ =	0.9575624671
Second flattening	n =	0.001679220386

Then	Latitude	φ	=	55°00'00.000"N
	Longitude	λ	=	10°00'00.000"E

1.3.14.1 Equidistant Cylindrical (Spherical)
(EPSG dataset coordinate operation method code 1029)

This method has one of the simplest formulas available. If the latitude of natural origin (φ_1) is at the equator the method is also known as Plate Carrée. It is not used for rigorous topographic mapping because its distortion characteristics are unsuitable. Formulas are included to distinguish this map projection method from an approach sometimes mistakenly called by the same name and used for simple computer display of geographic coordinates – see Pseudo Plate Carrée below.

For the forward calculation of the Equidistant Cylindrical method:

$$E = FE + R (\lambda - \lambda_O) \cos(\varphi_1)$$
$$N = FN + R \varphi$$

where φ_1, λ_O, φ and λ are expressed in radians.

R is the radius of the sphere and will normally be one of the CRS parameters. If the figure of the earth used is an ellipsoid rather than a sphere then R should be calculated as the radius of the conformal sphere at the projection origin at latitude φ_1 using the formula for R_C given in section 1.2, table 3. Note however that if applying spherical formula to ellipsoidal coordinates, the equidistant projection properties are not preserved.

For the reverse calculation:

$\varphi = (N - FN) / R$
$\lambda = \lambda_O + \{[E - FE] / [R \cos(\varphi_1)]\}$

where R is as for the forward method.

1.3.14.2 Pseudo Plate Carrée
(EPSG dataset coordinate operation method code 9825)

This is not a map projection in the true sense as the coordinate system units are angular (for example, decimal degrees) and therefore of variable linear scale. It is used only for depiction of graticule (latitude/longitude) coordinates on a computer display. The origin is at latitude (φ) = 0, longitude (λ) = 0. See above for the formulas for the proper Plate Carrée map projection method.

For the forward calculation:

$X = \lambda$
$Y = \varphi$

For the reverse calculation:

$\varphi = Y$
$\lambda = X$

1.3.15 Bonne
(EPSG dataset coordinate operation method code 9827)

The Bonne projection was frequently adopted for 18th and 19th century mapping, but being equal area rather than conformal its use for topographic mapping was replaced during the 20th century by conformal map projection methods.

The formulas to convert geodetic latitude and longitude (φ, λ) to Easting and Northing are:

$E = (\rho \sin T) + FE$
$N = (a\, m_O / \sin \varphi_O - \rho \cos T) + FN$

where
$m = \cos\varphi / (1 - e^2\sin^2\varphi)^{0.5}$
with φ in radians and m_O for φ_O, the latitude of the origin, derived in the same way.

$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024+....)\sin2\varphi$
$+ (15e^4/256 + 45e^6/1024 +.....)\sin4\varphi - (35e^6/3072 + ...)\sin6\varphi + ...]$
with φ in radians and M_O for φ_O, the latitude of the origin, derived in the same way.

$\rho = a\; m_O / \sin \varphi_O + M_O - M$
$T = a\; m (\lambda - \lambda_O) / \rho$ with λ and λ_O in radians

For the reverse calculation:
$X = E - FE$

$Y = N – FN$
$\rho = \pm [X^2 + (a . m_O / \sin \varphi_O – Y)^2]^{0.5}$ taking the sign of φ_O
$M = a\ m_O / \sin \varphi_O + M_O – \rho$
$\mu = M / [a (1 – e^2/4 – 3e^4/64 – 5e^6/256 – \ldots)]$
$e_1 = [1 – (1 – e^2)^{0.5}] / [1 + (1 – e^2)^{0.5}]$
$\varphi = \mu + (3e_1/2 – 27e_1^3/32 +.....)\sin2\mu + (21e_1^2/16 – 55e_1^4/32 + ...)\sin4\mu$
$+ (151e_1^3/96 +.....)\sin6\mu + (1097e_1^4/512 – ....)\sin8\mu + ...$
$m = \cos \varphi / (1 – e^2 \sin^2 \varphi)^{0.5}$

If φ_O is not negative
$\lambda = \lambda_O + \rho \{atan[X / (a . m_O / \sin \varphi_O – Y)]\} / a . m$
but if φ_O is negative
$\lambda = \lambda_O + \rho \{atan[– X / (Y – a . m_O / \sin \varphi_O)]\} / a . m$
In either case, if $\varphi = \pm 90°$, $m = 0$ and the equation for λ is indeterminate, so use $\lambda = \lambda_O$.

1.3.15.1 Bonne (South Orientated)

(EPSG dataset coordinate operation method code 9828)

In Portugal a special case of the method with coordinate system axes positive south and west has been used for older mapping. The formulas are as for the general case above except:

$W = FE – (\rho \ \sin T)$
$S = FN – (a\ m_O / \sin \varphi_O – \rho\ \cos T)$

In these formulas the terms FE and FN retain their definition, i.e. in the Bonne (South Orientated) method they increase the Westing and Southing value at the natural origin. In this method they are effectively false westing (FW) and false southing (FS) respectively.

For the reverse formulas, those for the standard Bonne method above apply, with the exception that:
$X = FE – W$
$Y = FN – S$

1.3.16 Azimuthal Equidistant

1.3.16.1 Modified Azimuthal Equidistant

(EPSG dataset coordinate operation method code 9832)

For various islands in Micronesia the US National Geodetic Survey has developed formulae for the oblique form of the ellipsoidal projection which calculates distance from the origin along a normal section rather than the geodesic. For the distances over which these projections are used (under 800km) this modification introduces no significant error.

First calculate a constant for the projection:
$\nu_O = a /(1 – e^2\sin^2\varphi_O)^{1/2}$

Then the forward conversion from latitude and longitude is given by:

$\nu = a /(1 – e^2\sin^2\varphi)^{1/2}$

$\psi = \text{atan} [(1 – e^2) \tan \varphi + e^2 \nu_O \sin \varphi_O / (\nu \cos \varphi)]$

$\alpha = \text{atan} \{\sin (\lambda – \lambda_O) / [\cos \varphi_O \tan \psi – \sin \varphi_O \cos (\lambda – \lambda_O)]\}$

$G = e \sin \varphi_O /(1 – e^2)^{1/2}$

$H = e \cos \varphi_O \cos \alpha /(1 – e^2)^{1/2}$

Then

if $(\sin \alpha) = 0$, $s = \text{asin} (\cos \varphi_O \sin \psi – \sin \varphi_O \cos \psi) * \text{SIGN}(\cos \alpha)$

if $(\sin \alpha) \neq 0$, $s = \text{asin} [\sin (\lambda – \lambda_O) \cos \psi / \sin \alpha]$

and in either case

$c = \nu_O s \{[1 – s^2 H^2 (1 – H^2) /6] + [(s^3/8)GH(1-2H^2)] + (s^4/120)[H^2(4-7H^2) – 3G^2(1-7H^2)] – [(s^5/48)GH]\}$

Then

$E = FE + c \sin \alpha$

$N = FN + c \cos \alpha$

For the reverse conversion from easting and northing to latitude and longitude:

$c' = [(E – FE)^2 + (N – FN)^2]^{0.5}$

$\alpha' = \text{atan} [(E – FE) / (N – FN)]$

$A = – e^2 \cos^2 \varphi_O \cos^2 \alpha' / (1 – e^2)$

$B = 3e^2 (1-A) \sin \varphi_O \cos \varphi_O \cos \alpha' / (1 – e^2)$

$D = c' / \nu_O$

$J = D – [A (1 + A) D^3 / 6] – [B (1 + 3A) D^4 / 24]$

$K = 1 – (A J^2 / 2) – (B J^3 / 6)$

$\Psi' = \text{asin} (\sin \varphi_O \cos J + \cos \varphi_O \sin J \cos \alpha')$

Then

$\varphi = \text{atan} [(1 – e^2 K \sin \varphi_O / \sin \Psi') \tan \Psi' / (1 – e^2)]$

$\lambda = \lambda_O + \text{asin} (\sin \alpha' \sin J / \cos \Psi')$

Example:

For Projected Coordinate Reference System: Guam 1963 / Yap Islands

Parameters:

Ellipsoid:	Clarke 1866	a = 6378206.400 metres	1/f = 294.97870
	then	e = 0.08227185	e^2 = 0.00676866

Latitude of natural origin	φ_O	9°32'48.15"N	=	0.166621493 rad
Longitude of natural origin	λ_O	138°10'07.48"E	=	2.411499514 rad
False easting	FE	40000.00 metres		
False northing	FN	60000.00 metres		

Forward calculation for:

Latitude	φ	=	9°35'47.493"N	=	0.167490973 rad
Longitude	λ	=	138°11'34.908"E	=	2.411923377 rad

First gives

ν_O =	6378800.24	G =	0.013691332
ν =	6378806.40	H =	0.073281276
ψ =	0.167485249	s =	0.000959566
α =	0.450640866	c =	6120.88

whence

E = 42665.90
N = 65509.82

Reverse calculation for the same Easting and Northing (42665.90 m E, 65509.82 m N) first gives:

c' =	6120.88	D =	0.000959566
α' =	0.450640866	J =	0.000959566
A =	-0.005370145	K =	1.000000002
B =	0.003026119	Ψ' =	0.167485249

whence

φ = 0.167490973 rad = 9°35'47.493"N
λ = 2.411923377 rad = 138°11'34.908"E

1.3.16.2 Guam Projection

(EPSG dataset coordinate operation method code 9831)

The Guam projection is a simplified form of the oblique case of the azimuthal equidistant projection. For the Guam projection the forward conversion from latitude and longitude is given by:

$$x = a\,(\lambda - \lambda_O) \cos\varphi \,/\, [(1 - e^2 \sin^2\varphi)^{(1/2)}]$$
$$E = FE + x$$
$$N = FN + M - M_O + \{x^2 \tan\varphi\, [(1 - e^2 \sin^2\varphi)^{(1/2)}] / (2a)\}$$

where

$$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024 + ...)\sin 2\varphi$$
$$+ (15e^4/256 + 45e^6/1024 + ...)\sin 4\varphi - (35e^6/3072 + ...)\sin 6\varphi + ...]$$

with φ in radians and M_O for φ_O, the latitude of the natural origin, derived in the same way.

The reverse conversion from easting and northing to latitude and longitude requires iteration of three equations. The Guam projection uses three iterations, which is satisfactory over the small area of application. First M_O for the latitude of the origin φ_O is derived as for the forward conversion. Then:

$$e_1 = [1 - (1 - e^2)^{0.5}] / [1 + (1 - e^2)^{0.5}]$$

and

$$M' = M_O + (N - FN) - \{(E - FE)^2 \tan\varphi_O\, [(1 - e^2 \sin^2\varphi_O)^{(1/2)}] / (2a)\}$$
$$\mu' = M' / a(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)$$
$$\varphi' = \mu' + (3e_1/2 - 27e_1^3/32)\sin(2\mu') + (21e_1^2/16 - 55e_1^4/32)\sin(4\mu') + (151e_1^3/96)\sin(6\mu')$$
$$+ (1097e_1^4/512)\sin(8\mu')$$

$$M'' = M_O + (N - FN) - \{(E - FE)^2 \tan\varphi'\, [(1 - e^2 \sin^2\varphi')^{(1/2)}] / (2a)\}$$
$$\mu'' = M'' / a(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)$$
$$\varphi'' = \mu'' + (3e_1/2 - 27e_1^3/32)\sin(2\mu'') + (21e_1^2/16 - 55e_1^4/32)\sin(4\mu'') + (151e_1^3/96)\sin(6\mu'')$$
$$+ (1097e_1^4/512)\sin(8\mu'')$$

$$M''' = M_O + (N - FN) - \{(E - FE)^2 \tan\varphi''\, [(1 - e^2 \sin^2\varphi'')^{(1/2)}] / (2a)\}$$
$$\mu''' = M''' / a(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)$$
$$\varphi''' = \mu''' + (3e_1/2 - 27e_1^3/32)\sin(2\mu''') + (21e_1^2/16 - 55e_1^4/32)\sin(4\mu''') + (151e_1^3/96)\sin(6\mu''')$$

$+ (1097e_1^4/512)\sin(8\mu''')$

Then

$\lambda = \lambda_O + \{(E - FE)\ [(1 - e^2 \sin^2\varphi''')^{(1/2)}] / (a \cos \varphi''')\}$

Example:

For Projected Coordinate Reference System: Guam 1963 / Guam SPCS

Parameters:

Ellipsoid:	Clarke 1866	a = 6378206.400 metres	1/f = 294.97870
	then	e = 0.08227185	e^2 = 0.00676866

Latitude of natural origin	φ_O	13°28'20.87887"N	=	0.235138896 rad
Longitude of natural origin	λ_O	144°44'55.50254"E	=	2.526342288 rad
False easting	FE	50000.00 metres		
False northing	FN	50000.00 metres		

Forward calculation for:

Latitude	φ	=	13°20'20.53846"N	=	0.232810140 rad
Longitude	λ	=	144°38'07.19265"E	=	2.524362746 rad

x	=	– 12287.52 m
M_O	=	1489888.76 m
M	=	1475127.96 m

whence

E	=	37,712.48 m
N	=	35,242.00 m

Reverse calculation for the same Easting and Northing (37,712.48 m E, 35,242.00 m N) first gives:

M_O	=	1489888.76 m
e_1	=	0.001697916

and

	M (metres)	μ (radians)	φ (radians)		
First iteration:	1475127.93	0.231668814	0.232810136		
Second iteration:	1475127.96	0.231668819	0.232810140		
Third iteration:	1475127.96	0.231668819	0.232810140	=	13°20'20.538"N

Then

λ = 2.524362746 rad = 144°38'07.193"E

1.3.17 Perspectives

1.3.17.1 Intoduction

Geophysical and reservoir interpretation and visualisation systems now work in a 3D "cube" offering continuous, scaleable, viewing and mapping in a single Cartesian 3D coordinate system. Subsurface mapping historically has been undertaken in pseudo-3D coordinate reference systems consisting of a vertical component together with an independent horizontal component which had to be changed to maintain cartographic correctness over large areas. Map projections are inherently distorted. Typically, distances and areas measured on the map-grid only approximate their true values. Over small areas near the projection origin, the distortions can be managed to be within acceptable limits. But it is impossible to map large areas without significant distortion. This creates problems when there is a requirement to map areas beyond the limits of a map zone, typically overcome by moving to another zone.

The motivation here is to offer a method of overcoming these limitations by describing geodetically well-defined CRSs that can be implemented in 3D within a visualisation environment and can be scaled (from reservoirs to regions) without distortion. There are three components:

- the use of geodetically rigorous 3D geocentric and topocentric coordinates, the relationship of which to geographical coordinates is descibed in section 2.2;
- perspective realizations of topocentric coordinates in 2D (sections 1.3.17.2 and 1.3.17.3);
- an ellipsoidal development of the orthographic projection (section 1.3.18). This 2D representation contains the quantifiable mapping distortions inherent in this projection method.

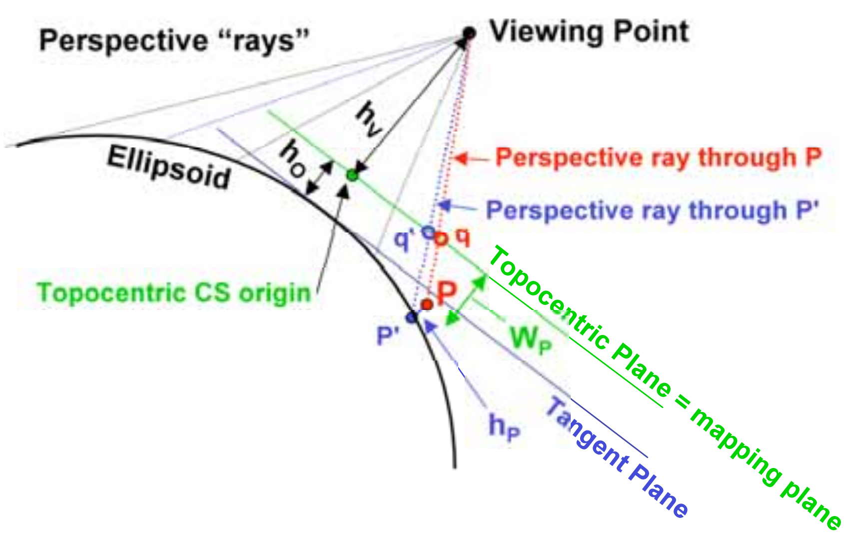

Figure 9. Vertical perspective

Classical map projections map 2D latitude and longitude onto the plane. With reference to figure 9 above, point P at a height h_P above the ellipsoid is first reduced to the ellipsoid surface at P', and P' is then mapped onto the mapping plane at q'. The height of the point is not material.

In contrast, perspectives map points on, above or below the surface of the ellipsoid onto the mapping plane; point P is mapped onto the mapping plane at q. The height of a point above or depth below the surface of the ellipsoid will change the horizontal coordinates at which the point maps. Perspectives are a view of the Earth from space without regard to cartographic properties such as conformality or equality of area.

Perspectives can be classified as vertical or tilted. Consider a point anywhere on the ellipsoid, a plane tangent to the ellipsoid at that point, and a perpendicular to the ellipsoid (and the tangent plane) at that point. Vertical perspectives are the view of the Earth from a point on the perpendicular through a mapping plane which is either the tangent plane or a plane parallel to the tangent plane. Tilted perspectives are the view from a point which is not on the perpendicular. Tilted perspectives are not considered further in this guidance note.

In addition to vertical and tilted, perspectives can be classified as positive or negative. Perspectives with a positive viewing height h_V are the view of the Earth from above, as from a satellite or from another celestial body (and as shown in figure 9). Perspectives with a negative viewing height h_V are the "view" of the Earth from below, which is mathematically but not physically possible. The mapping equations, however, are identical; only the sign of one term (the viewing height, h_V) differs. The viewing point cannot be on the mapping plane.

In this development vertical perspectives are based upon topocentric coordinates that are valid for an ellipsoidal Earth. The introduction of an intermediate topocentric coordinate system (see Section 2.2) simplifies the mathematical exposition of vertical perspectives. In such a topocentric Cartesian coordinate system, two of the three axes represent the horizontal plane. A change of perspective (zooming in and out) is achieved by moving the viewing point along the perpendicular. The mapping plane is the plane parallel to the tangent plane which passes through the topocentric origin (rather than the tangent plane itself). In the special case of the topocentric origin being on the ellipsoid surface then the mapping plane will be the tangent plane.

1.3.17.2 Vertical Perspective

(EPSG dataset coordinate operation method code 9838)

This general case deals with a viewing point at a finite height above the origin. If the viewing point is at infinity ($h_V = \infty$), the formulas for the orthographic case given in the next section should be used.

The forward equations for the Vertical Perspective to convert geographical 3D coordinates (φ, λ, h) to Easting (E) and Northing (N) begin with the methods of Section 2.2.3 to convert the geographical coordinates to topocentric coordinates U, V, W. The perspective projection origin is coincident with the topographic origin and has coordinates (φ_O, λ_O, h_O). As in Section 2.2.3:

$$U = (\nu + h) \cos \varphi \sin (\lambda - \lambda_O)$$
$$V = (\nu + h) [\sin \varphi \cos \varphi_O - \cos \varphi \sin \varphi_O \cos (\lambda - \lambda_O)] + e^2 (\nu_O \sin \varphi_O - \nu \sin \varphi) \cos \varphi_O$$
$$W = (\nu + h) [\sin \varphi \sin \varphi_O + \cos \varphi \cos \varphi_O \cos (\lambda - \lambda_O)] + e^2 (\nu_O \sin \varphi_O - \nu \sin \varphi) \sin \varphi_O - (\nu_O + h_O)$$

Then, given the height h_V of the perspective viewing point above the origin, the perspective coordinates (E, N) are calculated from topocentric coordinates (U, V, W) as:

$$E = U\, h_V / (h_V - W)$$
$$N = V\, h_V / (h_V - W)$$

The reverse calculation from E,N to U,V,W and φ,λ,h is indeterminate.

Example:

For Projected Coordinate Reference System: WGS 84 / Vertical Perspective example

Parameters:

Ellipsoid: WGS 84 a = 6378137.0 metres 1/f = 298.2572236
then e = 0.081819191

Topographic origin latitude	φ_O	=	55°00'00.000"N	=	0.95993109 rad
Topographic origin longitude	λ_O	=	5°00'00.000"E	=	0.08726646 rad
Topographic origin ellipsoidal height	h_O	=	200 metres		
Height of viewpoint	h_V	=	5 900 kilometers		

Forward calculation for:

Latitude	φ	=	53°48'33.82"N	=	0.939151101 rad
Longitude	λ	=	2°07'46.38"E	=	0.037167659 rad
Ellipsoidal height	h	=	73 metres		

e^2 = 0.006694380
ν_O = 6392 510.73 m
ν = 6392 088.02 m

U = –189 013.869 m
V = –128 642.040 m

$W = -4\,220.171$ m

Then,

$E = –188\,878.767$ m
$N = –128\,550.090$ m

1.3.17.3 Vertical Perspective (orthographic case)

(EPSG dataset coordinate operation method code 9839)

The orthographic vertical perspective is a special case of the vertical perspective with the viewing point at infinity ($h_V = \infty$). Therefore, all projection "rays" are parallel to one another and all are perpendicular to the tangent plane. Since the rays are parallel, coordinates in the tangent-plane are the same in any other parallel mapping plane, i.e. are consistent for any value of h_O, which therefore becomes irrelevant to the forward formulas.

The orthographic vertical perspective forward conversion from 3D geographic coordinates latitude, longitude and ellipsoidal height (φ, λ, h) to Easting (E) and Northing (N) is given by:

$$E = U = \text{limit } (U\, h_V / (h_V - W),\ h_V \rightarrow \infty)$$
$$N = V = \text{limit } (V\, h_V / (h_V - W),\ h_V \rightarrow \infty)$$

where, as in Sections 2.2.3 and 1.3.17.2:

$$U = (\nu + h) \cos\varphi \sin(\lambda - \lambda_O)$$
$$V = (\nu + h) [\sin\varphi \cos\varphi_O - \cos\varphi \sin\varphi_O \cos(\lambda - \lambda_O)] + e^2 (\nu_O \sin\varphi_O - \nu \sin\varphi) \cos\varphi_O$$

The reverse calculation from E,N to U,V,W and φ,λ,h is indeterminate.

Example:

For Projected Coordinate Reference System: WGS 84 / Vertical Perspective (Orthographic case) example

Parameters:

Ellipsoid: WGS 84 a = 6378137.0 metres 1/f = 298.2572236
then e = 0.081819191

Topographic origin latitude	φ_O	=	55°00'00.000"N	=	0.95993109 rad
Topographic origin longitude	λ_O	=	5°00'00.000"E	=	0.08726646 rad
Topographic origin ellipsoidal height	h_O	=	200 metres		

Forward calculation for:

Latitude	φ	=	53°48'33.82"N	=	0.939151101 rad
Longitude	λ	=	2°07'46.38"E	=	0.037167659 rad
Ellipsoidal height	h	=	73 metres		

The projection origin and example point are the same as those used in the general case of the Vertical Perspective in the previous section. Note that the ellipsoidal height at the point to be converted (h) is 73 metres. The ellipsoidal height at the topocentric center (h_O) is not used in any of the equations for the numerical examples that follow. But h_O will be used for the reverse case if W is known (for which a numerical example can be found in Section 2.2.3).

$e^2 = 0.006694380$
$\nu_O = 6392\,510.727$ m

$\nu = 6392\,088.017$ m

Then,

$E = -189013.869$ m
$N = -128642.040$ m

1.3.18 Orthographic Projection

(EPSG dataset coordinate operation method code 9840)

Most cartographic texts which describe the orthographic projection do so using a spherical development. This section describes an ellipsoidal development. This allows the projected coordinates to be consistent with those for the vertical perspectives described in the previous section (1.3.17). If the projection origin is at the topocentric origin, the ellipsoidal Orthographic Projection is a special case of the orthographic vertical perspective in which the ellipsoid height of all mapped points is zero (h = 0). The projection is neither conformal nor equal-area, but near the point of tangency there is no significant distortion. Within 90km of the origin the scale change is less than 1 part in 10,000.

The Orthographic Projection forward conversion from 2D geographic coordinates latitude and longitude (φ, λ) and the origin on the ellipsoid (φ_O, λ_O) is given by:

$$E = FE + \nu \cos\varphi \sin(\lambda - \lambda_O)$$
$$N = FN + \nu [\sin\varphi \cos\varphi_O - \cos\varphi \sin\varphi_O \cos(\lambda - \lambda_O)] + e^2 (\nu_O \sin\varphi_O - \nu \sin\varphi) \cos\varphi_O$$

where

ν is the prime vertical radius of curvature at latitude φ; $\nu = a/(1 - e^2\sin^2\varphi)^{0.5}$,
ν_O is the prime vertical radius of curvature at latitude of origin φ_O; $\nu_O = a/(1 - e^2\sin^2\varphi_O)^{0.5}$,
e is the eccentricity of the ellipsoid and $e^2 = (a^2 - b^2)/a^2 = 2f - f^2$
a and b are the ellipsoidal semi-major and semi-minor axes,
1/f is the inverse flattening, and
the latitude and longitude of the projection origin are φ_O and λ_O.

These formulas are similar to those for the orthographic case of the vertical perspective (section 1.3.17.3) except that, for the Orthographic Projection given here, h = 0 and the term (ν + h) reduces to ν. The projection origin is at the topocentric system origin φ_O, λ_O with false origin coordinates FE and FN.

For the reverse formulas for latitude and longitude corresponding to a given Easting (E) and Northing (N), iteration is required as the prime vertical radius (ν) is a function of latitude.

Begin by seeding the iteration with the center of projection (or some better guess):

$\varphi = \varphi_O$
$\lambda = \lambda_O$

Enter the iteration here with the (next) best estimates of φ and λ. Then solve for the radii of curvature in the prime vertical (ν) and meridian (ρ):

$\nu = a / (1 - e^2 \sin^2\varphi)^{0.5}$
$\rho = a (1 - e^2) / (1 - e^2 \sin^2\varphi)^{1.5}$

Compute test values of E and N (E' and N') using the forward equations:

$$E' = FE + \nu \cos\varphi \sin(\lambda - \lambda_O)$$
$$N' = FN + \nu [\sin\varphi \cos\varphi_O - \cos\varphi \sin\varphi_O \cos(\lambda - \lambda_O)] + e^2 (\nu_O \sin\varphi_O - \nu \sin\varphi) \cos\varphi_O$$

Partially differentiate the forward equations to solve for the elements of the Jacobian matrix:

$J_{11} = \partial E/\partial\varphi = -\rho \sin\varphi \sin(\lambda - \lambda_O)$
$J_{12} = \partial E/\partial\lambda = \nu \cos\varphi \cos(\lambda - \lambda_O)$
$J_{21} = \partial N/\partial\varphi = \rho(\cos\varphi \cos\varphi_O + \sin\varphi \sin\varphi_O \cos(\lambda - \lambda_O))$
$J_{22} = \partial N/\partial\lambda = \nu \sin\varphi_O \cos\varphi \sin(\lambda - \lambda_O)$

Solve for the determinant of the Jacobian:
$D = J_{11} J_{22} - J_{12} J_{21}$

Solve the northerly and easterly differences this iteration:
$\Delta E = E - E'$
$\Delta N = N - N'$

Adjust the latitude and longitude for the next iteration by inverting the Jacobian and multiplying by the differences:
$\varphi = \varphi + (J_{22}\, \Delta E - J_{12}\, \Delta N) / D$
$\lambda = \lambda + (-J_{21}\, \Delta E + J_{11}\, \Delta N) / D$

Return to the entry point with new estimates of latitude and longitude and iterate until the change in φ and λ is not significant.

Example:
For Projected Coordinate Reference System: WGS 84 / Orthographic Projection example

Parameters:

Ellipsoid:	WGS 84	a = 6378137.0 metres	1/f = 298.2572236
	then	e = 0.081819191	

Latitude of natural origin	φ_O	55°00'00.000"N	=	0.95993109 rad
Longitude of natural origin	λ_O	5°00'00.000"E	=	0.08726646 rad
False easting	FE	0 metres		
False northing	FN	0 metres		

Forward calculation for:

Latitude	φ	=	53°48'33.82"N	=	0.939151101 rad
Longitude	λ	=	2°07'46.38"E	=	0.037167659 rad

Note that ellipsoidal heights at the topocentric center (h_O) and at the point to be converted (h) may be the same as in the Vertical Perspective examples in the previous section. Neither enter the computations that follow.

$e^2 = 0.006694380$
$\nu_O = 6392\,510.73$ m
$\nu = 6392\,088.02$ m

Then,
Easting, E = –189 011.711 m
Northing, N = –128 640.567 m

Reverse calculation for these E, N coordinates into latitude (φ) and longitude (λ) is iterative. The following values are constant every iteration.

$e^2 = 0.006694380$
$\nu_O = 6392\,510.73$ m
$\varphi_O = 0.95993109$ rad

λ_O = 0.08726646 rad

The following values change during 4 iterations to convergence:

	1	2	3	4
Latitude	0.95993109	0.9397628327	0.9391516179	0.9391511016
Longitude	0.08726646	0.0357167858	0.0371688977	0.0371676590
ν	6392510.727	6392100.544	6392088.028	6392088.017
ρ	6378368.440	6377140.690	6377103.229	6377103.198
E'	0	–194318.490	–189006.908	–189011.711
N'	0	–124515.840	–128637.469	–128640.567
J_{11}	0	265312.746	257728.730	257734.999
J_{12}	3666593.522	3766198.868	3769619.566	3769621.986
J_{21}	6378368.440	6370240.831	6370437.125	6370436.766
J_{22}	0	–159176.388	–154825.395	–154829.329
D	–2338688440386	–24033825331760	–24054027385585	–24054043431047
ΔN	–128640.567	–4124.727	–3.098	0
ΔE	–189011.711	5306.779	–4.803	0
Latitude	0.9397628327	0.9391516179	0.9391511016	0.9391511016
Longitude	0.0357167858	0.0371688977	0.0371676590	0.0371676590

which results in:

Latitude	φ	=	0.939151102 rad	=	53°48'33.82"N
Longitude	λ	=	0.037167659 rad	=	2°07'46.38"E

2 Formulas for Coordinate Operations other than Map Projections

2.1 Introduction

Several types of coordinate reference system are recognised. The previous section discussed conversions of coordinates between geographic 2-dimensional and projected coordinate reference systems. The projected system is derived from its base geographic system.

Geographical coordinates (latitude and longitude) are calculated on a model of the earth. They are only unique and unambiguous when the model and its relationship to the real earth is identified. This is accomplished through a geodetic datum. A change of geodetic datum changes the geographic coordinates of a point. A geodetic datum combined with description of coordinate system gives a coordinate reference system. Coordinates are only unambiguous when their coordinate reference system is identified and defined.

It is frequently required to change coordinates derived in one geographic coordinate reference system to values expressed in another. For example, land and marine seismic surveys are nowadays most conveniently positioned by GPS satellite in the WGS 84 geographic coordinate reference system, whereas coordinates may be required referenced to the national geodetic reference system in use for the country concerned. It may therefore be necessary to transform the observed WGS 84 data to the national geodetic reference system in order to avoid discrepancies caused by the change of geodetic datum.

Some transformation methods operate directly between geographic coordinates. Others are between geocentric coordinates (3-dimensional Cartesian coordinates where the coordinate system origin is fixed at the centre of the earth). The second part of this Guidance Note covers conversions and transformations between geographic coordinate reference systems, both directly and indirectly through geocentric systems. Some of these methods (polynomial family) may also be encountered for use between other types of coordinate reference systems, for example directly between projected coordinate reference systems. This second part also describes transformations of vertical coordinates.

Coordinate handling software may execute more complicated operations, concatenating a number of steps linking together geographic, projected and/or engineering coordinates referenced to different datums. Other than as mentioned above, these concatenated operations are beyond the scope of this document.

Note that it is very important to ensure that the signs of the parameter values used in the transformations are correct in respect of the transformation being executed. Preferably one should always express transformations in terms of "From".........."To"........... thus avoiding the confusion which may result from interpreting a dash as a minus sign or vice versa.

2.2 Coordinate Conversions other than Map Projections

2.2.1 Geographic/Geocentric conversions

(EPSG datset coordinate operation method code 9602)

Latitude, **φ,** and Longitude, **λ**, and ellipsoidal height, **h**, in terms of a 3-dimensional geographic coordinate reference system may be expressed in terms of a geocentric (earth centred) Cartesian coordinate reference system X, Y, Z with the Z axis corresponding with the earth's rotation axis positive northwards, the X axis through the intersection of the prime meridian and equator, and the Y axis through the intersection of the equator with longitude 90°E. The geographic and geocentric systems are based on the same geodetic datum.

Geocentric coordinate reference systems are conventionally taken to be defined with the X axis through the intersection of the Greenwich meridian and equator. This requires that the equivalent geographic coordinate reference system be based on the Greenwich meridian. In application of the formulas below, geographic coordinate reference systems based on a non-Greenwich prime meridian should first be transformed to their Greenwich equivalent. Geocentric coordinates X, Y and Z take their units from the units for the ellipsod axes (a and b). As it is conventional for X, Y and Z to be in metres, if the ellipsoid axis dimensions are given in another linear unit they should first be converted to metres.

If the ellipsoidal semi major axis is **a,** semi minor axis **b**, and inverse flattening **1/f**, then

$$X = (\nu + h) \cos\varphi \cos\lambda$$
$$Y = (\nu + h) \cos\varphi \sin\lambda$$
$$Z = [(1 - e^2)\,\nu + h] \sin\varphi$$

where ν is the prime vertical radius of curvature at latitude φ and is equal to
$\nu = a / (1 - e^2 \sin^2\varphi)^{0.5}$,
φ and λ are respectively the latitude and longitude (related to the prime meridian) of the point,
h is height above the ellipsoid, (see note below), and
e is the eccentricity of the ellipsoid where $e^2 = (a^2 - b^2)/a^2 = 2f - f^2$

(Note that h *is the height above the ellipsoid.. This is the height value that is delivered by GPS satellite observations but is not the gravity-related height value which is normally used for national mapping and levelling operations. The gravity-related height (*H*) is usually the height above mean sea level or an alternative level reference for the country. If one starts with a gravity-related height* H*, it will be necessary to convert it to an ellipsoid height* (h) *before using the above transformation formulas.* See section 2.4.5 below. *For the WGS 84 ellipsoid the difference between ellipsoid and mean sea level can vary between values of -100m in the Sri Lanka area to +80m in the North Atlantic.)*

For the reverse conversion, Cartesian coordinates in the geocentric coordinate reference system may be converted to geographic coordinates in terms of the geographic 3D coordinate reference system by:

$$\varphi = \operatorname{atan}\,(Z + e^2 \nu \sin\varphi) / (X^2 + Y^2)^{0.5} \text{ by iteration}$$
$$\lambda = \operatorname{atan} Y/X$$
$$h = X \sec\lambda \sec\varphi - \nu$$

where λ is relative to the Greenwich prime meridian.
To avoid iteration for φ it may alternatively be found from:

$$\varphi = \operatorname{atan}[(Z + \varepsilon\, b \sin^3 q) / (p - e^2 a \cos^3 q)]$$

where

$$\varepsilon = e^2 / (1 - e^2)$$
$$b = a(1 - f)$$
$$p = (X^2 + Y^2)^{0.5}$$
$$q = \operatorname{atan}[(Z\, a) / (p\, b)]$$

Then h may more conveniently be found from

$h = (p / \cos \varphi) - \nu$

Example:
Consider a North Sea point with position derived by GPS satellite in the WGS 84 coordinate reference system. The WGS 84 ellipsoid parameters are:

a	=	6378 137.000m
1/f	=	298.2572572236

then

e^2	=	0.006694380
ε	=	0.006739497
b	=	6356 752.314 m

Using the reverse direction direct formulas above, the conversion of WGS 84 geocentric coordinates of

X	=	3771 793.968 m
Y	=	140 253.342 m
Z	=	5124 304.349 m

is:

p	=	3774400.712
q	=	0.937546077
φ	=	0.939151101 rad
ν	=	6392088.017

Then WGS 84 geographic 3D coordinates are:

	latitude φ	=	53°48'33.820" N
	longitude λ	=	2°07'46.380"E
and	ellipsoidal height h	=	73.0m

2.2.2 Geocentric/topocentric conversions

(EPSG dataset coordinate operation method code 9836)

A topocentric coordinate system is a 3-D Cartesian system having mutually perpendicular axes U, V, W with an origin on or near the surface of the Earth. The U-axis is locally east, the V-axis is locally north and the W-axis is up forming a right-handed coordinate system. It is applied in two particular settings:

(i) the height axis W is chosen to be along the direction of gravity at the topocentric origin. The other two axes are then in the horizontal plane. A special case of this, often applied in engineering applications, is when the topocentric origin is on the vertical datum surface; then topocentric height W approximates to gravity-related height H.

(ii) the topocentric height axis W is chosen to be the direction through the topocentric origin and along perpendicular to the surface of the ellipsoid. The other two topocentic axes (U and V) are in the "topocentric plane", a plane parallel to the tangent to the ellipsoid surface at the topocentric origin and passing through the topocentric origin (see figure 11 below). The coordinates defining the topocentric origin will usually be expressed in ellipsoidal terms as latitude φ_O, longitude λ_O and ellipsoidal height h_O but may alternatively be expressed as geocentric Cartesian coordinates X_O, Y_O, Z_O. In this context the geocentric coordinates of the topocentric origin should not be confused with those of the geocentric origin where X=Y=Z=0.

A special case of this is when the topocentric origin is chosen to be exactly on the ellipsoid surface and h_O = 0. Then the topocentric U and V axes are in the ellipsoid tangent plane and at (and only at) the topocentric origin topocentric height W = ellipsoidal height h.

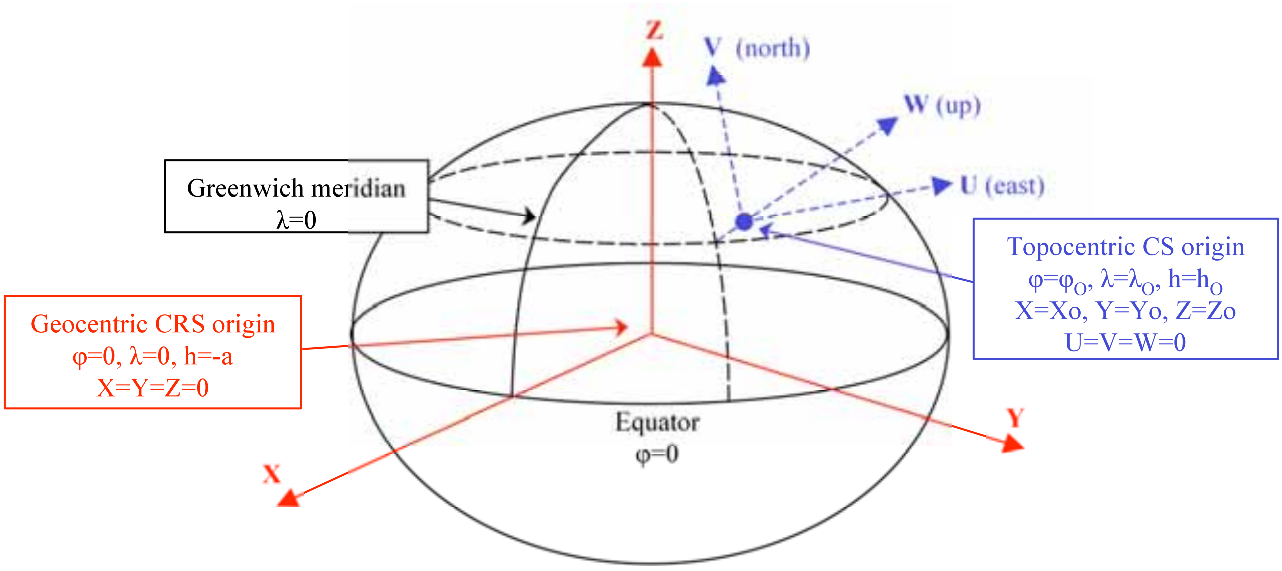

Figure 10. Topocentric and geocentric systems

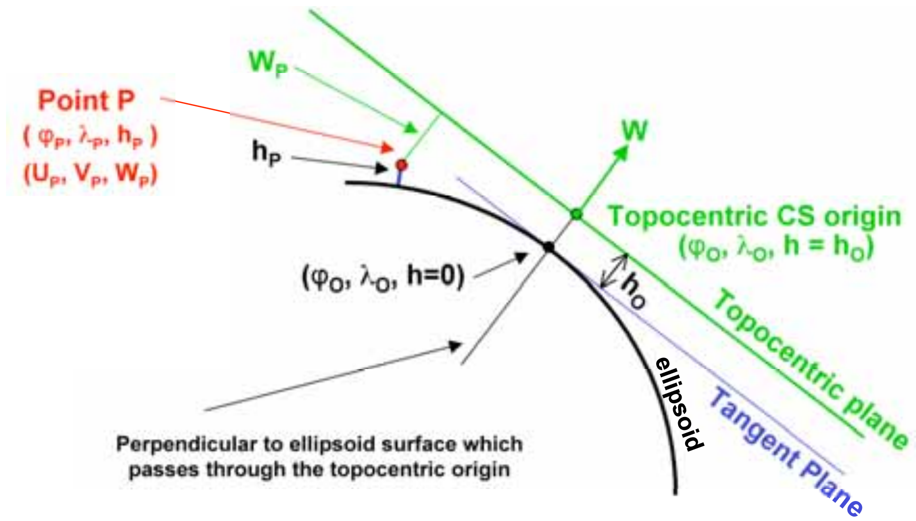

Figure 11. Topocentric and ellipsoidal heights

In this and the following section we are concerned with the second of the two settings for topocentric coordinate systems where the system is associated with the ellipsoid and a particular geodetic datum. The application of such topocentric coordinates includes scalable mapping and visualisation systems as described in section 1.3.17. The following section covers the conversion between ellipsoidal coordinates and topocentric coordinates. The remainder of this section describes how geocentric coordinates X, Y, Z may be converted into topocentric coordinates U, V, W given the geocentric coordinates of the topocentric CS origin (X_O, Y_O, Z_O).

First it is necessary to derive ellipsoidal values φ_O, λ_O of the topocentric origin from their geocentric values X_O, Y_O, Z_O through the reverse formulas given in Section 2.2.1 above. (The value h_O for the ellipsoidal height of the topocentric origin is not required in what follows.)

Then topocentric coordinates [U, V, W] are computed as follows:

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \boldsymbol{R} * \left(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} - \begin{pmatrix} X_O \\ Y_O \\ Z_O \end{pmatrix} \right)$$

where,

$$\boldsymbol{R} = \begin{pmatrix} -\sin\lambda_O & \cos\lambda_O & 0 \\ -\sin\varphi_O\cos\lambda_O & -\sin\varphi_O\sin\lambda_O & \cos\varphi_O \\ \cos\varphi_O\cos\lambda_O & \cos\varphi_O\sin\lambda_O & \sin\varphi_O \end{pmatrix}$$

Or, expressed as scalar equations:

$U = -(X-X_O)\sin\lambda_O + (Y-Y_O)\cos\lambda_O$
$V = -(X-X_O)\sin\varphi_O\cos\lambda_O - (Y-Y_O)\sin\varphi_O\sin\lambda_O + (Z-Z_O)\cos\varphi_O$
$W = (X-X_O)\cos\varphi_O\cos\lambda_O + (Y-Y_O)\cos\varphi_O\sin\lambda_O + (Z-Z_O)\sin\varphi_O$

The reverse formulas to calculate geocentric coordinates from topocentric coordinates are:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \boldsymbol{R}^{-1} * \begin{pmatrix} U \\ V \\ W \end{pmatrix} + \begin{pmatrix} X_O \\ Y_O \\ Z_O \end{pmatrix}$$

where,

$$\boldsymbol{R}^{-1} = \boldsymbol{R}^{T} = \begin{pmatrix} -\sin\lambda_0 & -\sin\varphi_0\cos\lambda_0 & \cos\varphi_0\cos\lambda_0 \\ \cos\lambda_0 & -\sin\varphi_0\sin\lambda_0 & \cos\varphi_0\sin\lambda_0 \\ 0 & \cos\varphi_0 & \sin\varphi_0 \end{pmatrix}$$

and, as for the forward case, φ_O and λ_O are calculated through the formulas in Section 2.2.1.

Or, expressed as scalar equations:

$X = X_O - U\sin\lambda_O - V\sin\varphi_O\cos\lambda_O + W\cos\varphi_O\cos\lambda_O$
$Y = Y_O + U\cos\lambda_o - V\sin\varphi_O\sin\lambda_O + W\cos\varphi_O\sin\lambda_O$
$Z = Z_O + V\cos\varphi_O + W\sin\varphi_O$

Example:

For Geocentric CRS = WGS 84 (EPSG CRS code 4978)
and
Topocentric origin Xo = 3652 755.3058 m
Topocentric origin Yo = 319 574.6799 m
Topocentric origin Zo = 5201 547.3536 m

Ellipsoid parameters: a = 6378137.0 metres 1/f = 298.2572236

First calculate additional ellipsoid parameters:

e^2 = 0.006694380 ε = 0.006739497 b = 6356752.314

Next, derive φ_O, λ_O from Xo,Yo,Zo by the formulas given in Section 2.2.1:

p = 3666708.2376
q = 0.9583523313
φ_O = 0.9599310885 rad
λ_O = 0.0872664625 rad

Forward calculation for point with geocentric coordinates:

X = 3771 793.968 m Y = 140 253.342 m Z = 5124 304.349 m

gives topocentric coordinates

U = –189 013.869 m V = –128 642.040 m W = – 4 220.171 m

The reverse calculation contains no intermediate terms other than those solved for above and is a trivial reversal of the forward.

2.2.3 <u>Geographic/topocentric conversions</u>

(EPSG dataset coordinate operation method code 9837)

Topocentric coordinates may be derived from geographic coordinates indirectly by concatenating the geographic/geocentric conversion described in 2.2.1 above with the geocentric/topocentric conversion described in 2.2.2 above. Alternatively the conversion may be made directly:

To convert latitude φ, longitude λ and ellipsoidal height h into topocentric coordinates U,V,W:

$$U = (\nu + h) \cos\varphi \sin(\lambda - \lambda_O)$$
$$V = (\nu + h) [\sin\varphi \cos\varphi_O - \cos\varphi \sin\varphi_O \cos(\lambda - \lambda_O)] + e^2 (\nu_O \sin\varphi_O - \nu \sin\varphi) \cos\varphi_O$$
$$W = (\nu + h) [\sin\varphi \sin\varphi_O + \cos\varphi \cos\varphi_O \cos(\lambda - \lambda_O)] + e^2 (\nu_O \sin\varphi_O - \nu \sin\varphi) \sin\varphi_O - (\nu_O + h_O)$$

where φ_O, λ_O, h_O are the ellipsoidal coordinates of the topocentric origin

and ν is the radius of curvature in the prime vertical at latitude $\varphi = a /(1 - e^2\sin^2\varphi)^{0.5}$
ν_O is the radius of curvature in the prime vertical at latitude $\varphi_O = a /(1 - e^2\sin^2\varphi_O)^{0.5}$
e is the eccentricity of the ellipsoid where $e^2 = (a^2 - b^2)/a^2 = 2f - f^2$

The reverse formulae to convert topocentric coordinates (U, V, W) into latitude, longitude and ellipsoidal height (φ, λ, h) first draws on the reverse case of section 2.2.2 to derive geocentric coordinates X, Y, Z and then on the reverse case in section 2.2.1 to derive latitude, longitude and height.

First,

$X = X_O – U \sin \lambda_O – V \sin \varphi_O \cos \lambda_O + W \cos \varphi_O \cos \lambda_O$
$Y = Y_O + U \cos \lambda_O – V \sin \varphi_O \sin \lambda_O + W \cos \varphi_O \sin \lambda_O$
$Z = Z_O + V \cos \varphi_O + W \sin \varphi_O$

where,

$X_O = (\nu_O + h_O) \cos \varphi_O \cos \lambda_O$
$Y_O = (\nu_O + h_O) \cos \varphi_O \sin \lambda_O$
$Z_O = [(1 – e^2) \nu_O + h_O]\sin \varphi_O$
φ_O, λ_O, h_O are the ellipsoidal coordinates of the topocentric origin,
ν_O is the radius of curvature in the prime vertical at latitude $\varphi_O = a /(1 – e^2\sin^2\varphi_O)^{0.5}$, and
e is the eccentricity of the ellipsoid where $e^2 = (a^2 – b^2)/a^2 = 2f – f^2$.

Then,

$\varphi = \text{atan}[(Z + \varepsilon\, b \sin^3 q) / (p – e^2\, a \cos^3 q)]$
$\lambda = \text{atan } Y/X$

where

$\varepsilon = e^2 / (1 – e^2)$
$b = a(1 – f)$
$p = (X^2 + Y^2)^{0.5}$
$q = \text{atan}[(Z\, a) / (p\, b)]$
λ is relative to the Greenwich prime meridian.

and

$h = (p / \cos \varphi) – \nu$

where

ν is the radius of curvature in the prime vertical at latitude $\varphi = a /(1 – e^2\sin^2\varphi)^{0.5}$

Example:

For Geographic 3D CRS = WGS 84 (EPSG CRS code 4979)
and

Topocentric origin latitude	φ_O	55°00'00.000"N	=	0.95993109 rad
Topocentric origin longitude	λ_O	5°00'00.000"E	=	0.08726646 rad
Topocentric origin ellipsoidal height	h_O	200 metres		

Ellipsoid parameters: a = 6378137.0 metres 1/f = 298.2572236

First calculate additional ellipsoid parameter e^2 and radius of curvature ν_O at the topocentric origin:

e^2 = 0.006694380 ν_O = 6392510.727

Forward calculation for:

Latitude	φ	=	53°48'33.82"N	=	0.93915110 rad
Longitude	λ	=	2°07'46.38"E	=	0.03716765 rad
Height	h	=	73.0 metres		

ν = 6392088.017

then

U = –189 013.869 m
V = –128 642.040 m
W = – 4 220.171 m

Reverse calculation for:

U = –189 013.869 m
V = –128 642.040 m
W = – 4 220.171 m

First calculate additional ellipsoid parameter e^2 and radius of curvature ν_O at the topocentric origin:

e^2 = 0.006694380 ν_O = 6392510.727

then the following intermediate terms:

X_O	=	3652 755.306	ε	=	0.0067394967
Y_O	=	319 574.680	b	=	6356 752.314
Z_O	=	5201 547.353	p	=	3774 400.712
			q	=	0.937549875
X	=	3771 793.968	φ	=	0.9391511015 rad
Y	=	140 253.342	ν	=	6392 088.017
Z	=	5124 304.349	λ	=	0.03716765908 rad

for a final result of:

Latitude	φ	=	53°48'33.820"N
Longitude	λ	=	2°07'46.380"E
Height	h	=	73.0 metres

2.2.4 Geographic 3D to 2D conversions

(EPSG dataset coordinate operation method code 9659)

The forward case is trivial. A 3-dimensional geographic coordinate reference system comprising of geodetic latitude, geodetic longitude and ellipsoidal height is converted to its 2-dimensional subset by the simple expedient of dropping the height.

The reverse conversion, from 2D to 3D, is indeterminate. It is however a requirement when a geographic 2D coordinate reference system is to be transformed using a geocentric method which is 3-dimensional (see section 2.4.4.1 below). In practice an artificial ellipsoidal height is created and appended to the geographic 2D coordinate reference system to create a geographic 3D coordinate reference system referenced to the same geodetic datum. The assumed ellipsoidal height is usually either set to the gravity-related height of a position in a compound coordinate reference system, or set to zero. As long as the height chosen is within a few kilometres of sea level, no error will be induced into the horizontal position resulting from the later geocentric transformation; the vertical coordinate will however be meaningless.

Example:

A location in the ETRS89 Geographic 3D coordinate reference system

latitude φ_s = 53°48'33.82"N
longitude λ_s = 2°07'46.38"E
and ellipsoidal height h_s = 73.0m

is converted to the ETRS89 Geographic 2D coordinate reference system as

latitude φ_s = 53°48'33.82"N
longitude λ_s = 2°07'46.38"E

For the reverse conversion of the same point in the ETRS89 Geographic 2D coordinate reference system with horizontal coordinates of

latitude φ_s = 53°48'33.82"N
longitude λ_s = 2°07'46.38"E

an arbitary value is given to the ellipsoidal height resulting in coordinates in the ETRS89 Geographic 3D coordinate reference system of

latitude φ_s = 53°48'33.82"N
longitude λ_s = 2°07'46.38"E
and ellipsoidal height h_s = 0.0m

2.3 Coordinate Operation Methods that can be conversions or transformations

In theory, certain coordinate operation methods do not readily fit the ISO 19111 classification of being either a coordinate conversion (no change of datum involved) or a coordinate transformation. These methods change coordinates directly from one coordinate reference system to another and may be applied with or without change of datum, depending upon whether the source and target coordinate reference systems are based on the same or different datums. In practice, most usage of these methods does in fact include a change of datum. OGP follows the general mathematical usage of these methods and classifies them as transformations.

2.3.1 Polynomial transformations

Note: In the sections that follow, the general mathematical symbols X and Y representing the axes of a coordinate reference system must not be confused with the specific axis abbreviations or axis order in particular coordinate reference systems.

2.3.1.1 General case

Polynomial transformations between two coordinate reference systems are typically applied in cases where one or both of the coordinate reference systems exhibits lack of homogeneity in orientation and scale. The small distortions are then approximated by polynomial functions in latitude and longitude or in easting and northing. Depending on the degree of variability in the distortions, approximation may be carried out using polynomials of degree 2, 3, or higher. In the case of transformations between two projected coordinate reference systems, the additional distortions resulting from the application of two map projections and a datum transformation can be included in a single polynomial approximation function.

Polynomial approximation functions themselves are subject to variations, as different approximation characteristics may be achieved by different polynomial functions. The simplest of all polynomials is the general polynomial function. In order to avoid problems of numerical instability this type of polynomial should be used after reducing the coordinate values in both the source and the target coordinate reference system to 'manageable' numbers, between –10 and +10 at most. This is achieved by working with offsets relative to a central evaluation point, scaled to the desired number range by applying a scaling factor to the coordinate offsets.

Hence an evaluation point is chosen in the source coordinate reference system (X_{S0}, Y_{S0}) and in the target coordinate reference system (X_{T0}, Y_{T0}). Often these two sets of coordinates do not refer to the same physical point but two points are chosen that have the same coordinate values in both the source and the target coordinate reference system. (When the two points have identical coordinates, these parameters are conveniently eliminated from the formulas, but the general case where the coordinates differ is given here).

The selection of an evaluation point in each of the two coordinate reference systems allows the point coordinates in both to be reduced as follows:

$X_S – X_{S0}$
$Y_S – Y_{S0}$

and

$X_T – X_{T0}$

$Y_T - Y_{T0}$

These coordinate differences are expressed in their own unit of measure, which may not be the same as that of the corresponding coordinate reference system.[5)]

A further reduction step is usually necessary to bring these coordinate differences into the desired numerical range by applying a scaling factor to the coordinate differences in order to reduce them to a value range that may be applied to the polynomial formulae below without introducing numerical precision errors:

$$U = m_S (X_S - X_{S0})$$
$$V = m_S (Y_S - Y_{S0})$$

where

X_S , Y_S are coordinates in the source coordinate reference system,
X_{S0} , Y_{S0} are coordinates of the evaluation point in the source coordinate reference system,
m_S is the scaling factor applied the coordinate differences in the source coordinate reference system.

The normalised coordinates U and V of the point whose coordinates are to be transformed are used as input to the polynomial transformation formula. In order to control the numerical range of the polynomial coefficients A_n and B_n the output coordinate differences dX and dY are multiplied by a scaling factor, m_T.

$$\begin{aligned}
m_T.dX \;=\; & A_0 + A_1U + A_2V + A_3U^2 + A_4UV + A_5V^2 && \text{(to degree 2)}\\
+\; & A_6U^3 + A_7U^2V + A_8UV^2 + A_9V^3 && \text{(degree 3 terms)}\\
+\; & A_{10}U^4 + A_{11}U^3V + A_{12}U^2V^2 + A_{13}UV^3 + A_{14}V^4 && \text{(degree 4 terms)}\\
+\; & A_{15}U^5 + A_{16}U^4V + A_{17}U^3V^2 + A_{18}U^2V^3 + A_{19}UV^4 + A_{20}V^5 && \text{(degree 5 terms)}\\
+\; & A_{21}U^6 + A_{22}U^5V + A_{23}U^4V^2 + A_{24}U^3V^3 + A_{25}U^2V^4 + A_{26}UV^5 + A_{27}V^6 && \text{(degree 6 terms)}\\
+\; & \ldots + A_{104}V^{13} && \text{(degree 13 terms)}
\end{aligned}$$

$$\begin{aligned}
m_T.dY \;=\; & B_0 + B_1U + B_2V + B_3U^2 + B_4UV + B_5V^2 && \text{(to degree 2)}\\
+\; & B_6U^3 + B_7U^2V + B_8UV^2 + B_9V^3 && \text{(degree 3 terms)}\\
+\; & B_{10}U^4 + B_{11}U^3V + B_{12}U^2V^2 + B_{13}UV^3 + B_{14}V^4 && \text{(degree 4 terms)}\\
+\; & B_{15}U^5 + B_{16}U^4V + B_{17}U^3V^2 + B_{18}U^2V^3 + B_{19}UV^4 + B_{20}V^5 && \text{(degree 5 terms)}\\
+\; & B_{21}U^6 + B_{22}U^5V + B_{23}U^4V^2 + B_{24}U^3V^3 + B_{25}U^2V^4 + B_{26}UV^5 + B_{27}V^6 && \text{(degree 6 terms)}\\
+\; & \ldots + B_{104}V^{13} && \text{(degree 13 terms)}
\end{aligned}$$

from which dX and dY are evaluated. These will be in the units of the target coordinate reference system.

In the EPSG dataset, the polynomial coefficients are given as parameters of the form Aumvn and Bumvn, where m is the power to which U is raised and n is the power to which V is raised. For example, A_{17} is represented as coordinate operation parameter Au3v2.

The relationship between the two coordinate reference systems can now be written as follows:

$$(X_T - X_{TO}) = (X_S - X_{SO}) + dX$$
$$(Y_T - Y_{TO}) = (Y_S - Y_{SO}) + dY$$

or

$$X_T = X_S - X_{SO} + X_{TO} + dX$$
$$Y_T = Y_S - Y_{SO} + Y_{TO} + dY$$

where:

X_T , Y_T are coordinates in the target coordinate reference system,

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5) If the source and/or the target coordinate reference system are geographical, the coordinates themselves may be expressed in sexagesimal degrees (degrees, minutes, seconds), which cannot be directly processed by a mathematical formula.

X_S , Y_S are coordinates in the source coordinate reference system,
X_{SO} , Y_{SO} are coordinates of the evaluation point in the source coordinate reference system,
X_{TO} , Y_{TO} are coordinates of the evaluation point in the target coordinate reference system,
dX, dY are derived through the scaled polynomial formulas.

Other (arguably better) approximating polynomials are described in mathematical textbooks such as *"Theory and applications of numerical analysis"*, by G.M. Phillips and P.J. Taylor (Academic Press, 1973).

Example: General polynomial of degree 6 (EPSG dataset coordinate operation method code 9648)
For coordinate transformation TM75 to ETRS89 (1)

Ordinate 1 of evaluation point X_O in source CRS: $X_{SO} = \varphi_{SO}$ = 53°30'00.000"N = +53.5 degrees
Ordinate 2 of evaluation point Y_O in source CRS: $Y_{SO} = \lambda_{SO}$ = 7°42'00.000"W = – 7.7 degrees
Ordinate 1 of evaluation point X_O in target CRS : $X_{TO} = \varphi_{TO}$ = 53°30'00.000"N = +53.5 degrees
Ordinate 2 of evaluation point Y_O in target CRS : $Y_{SO} = \lambda_{TO}$ = 7°42'00.000"W = – 7.7 degrees
Scaling factor for source CRS coordinate differences: $m_S = 0.1$
Scaling factor for target CRS coordinate differences: $m_T = 3600$

Coefficients (see EPSG dataset transformation code 1041 for complete set of values):
$A_0 = 0.763$ $A_1 = -4.487$ …. $A_{24} = -265.898$ ... $A_{27} = 0$
$B_0 = -2.810$ $B_1 = -0.341$ …. $B_{24} = -853.950$ ... $B_{27} = 0$

Forward calculation for:
Latitude $\varphi_{TM75} = X_s =$ 55°00'00"N = + 55.000 degrees
Longitude $\lambda_{TM75} = Y_s =$ 6°30'00"W = – 6.500 degrees

$X_S - X_{SO} = \varphi_{TM75} - \varphi_{S0} = 55.0 - 53.5 = 1.5$ degrees
$Y_S - Y_{SO} = \lambda_{TM75} - \lambda_{S0} = -6.5 - (-7.7) = 1.2$ degrees

$U = m_S (X_S - X_{S0}) = m_S (\varphi_{TM75} - \varphi_{S0}) = 0.1*(1.5) = 0.15$
$V = m_S (Y_S - Y_{S0}) = m_S (\lambda_{TM75} - \lambda_{S0}) = 0.1*(1.2) = 0.12$

$$dX = (A_0 + A_1U + ... + A_5V^2 + ... + A_{24}U^3V^3) / m_T$$
$$= [0.763 + (-4.487 * 0.15) + ... + (0.183 * 0.12^2) + ... + (-265.898 * 0.15^3 * 0.12^3)] / 3600$$

$$dY = (B_0 + B_1U + ... + B_{24}U^3V^3) / m_T$$
$$= [-2.81 + (-0.341 * 0.15) + ... + (-853.95 * 0.15^3 * 0.12^3)] / 3600$$

	to degree 2	degree 3	degree 4	degree 5	degree 6	Sum / m_T
dX =	0.1029127	-0.002185407	0.0064009440	0.0014247770	-0.0015507171	0.0000297229
dY =	-3.3955340	0.022364019	-0.0230149836	-0.0156886729	-0.0049802364	-0.0009491261

Then Latitude $\varphi_{ETRS89} = X_T = X_S + dX = 55.0 + 0.00002972$ degrees = 55°00'00.107"N
Longitude $\lambda_{ETRS89} = Y_T = Y_S + dY = -6.5 - 0.00094913$ degrees = 6°30'03.417"W

Polynomial reversibility
Approximation polynomials are in a strict mathematical sense ***not reversible***, i.e. the same polynomial coefficients cannot be used to execute the reverse transformation.

In principle two options are available to execute the reverse transformation:
1. By applying a similar polynomial transformation with a different set of polynomial coefficients for

the reverse polynomial transformation. This would result in a separate forward and reverse transformation being stored in the EPSG dataset (or any other geodetic data repository).

2. By applying the polynomial transformation with the same coefficients but with their signs reversed and then iterate to an acceptable solution, the number of iteration steps being dependent on the desired accuracy. (Note that only the signs of the polynomial coefficients should be reversed and not the coordinates of the evaluation points or the scaling factors!) The iteration procedure is usually described by the information source of the polynomial transformation.

However, under certain conditions, described below, a satisfactory solution for the reverse transformation may be obtained using the forward coefficient values in a single step, rather than multiple step iteration. If such a solution is possible, in the EPSG dataset the polynomial coordinate transformation method is classified as a ***reversible polynomial of degree n***.

A (general) polynomial transformation is reversible only when the following conditions are met.

1. The co-ordinates of source and target evaluation point are (numerically) the same.
2. The unit of measure of the coordinate differences in source and target coordinate reference system are the same.
3. The scaling factors applied to source and target coordinate differences are the same.
4. The spatial variation of the differences between the coordinate reference systems around any given location is sufficiently small.

Clarification on conditions for polynomial reversibility:

Re 1 and 2 - In the reverse transformation the roles of the source and target coordinate reference systems are reversed. Consequently, the co-ordinates of the evaluation point in the *source* coordinate reference system become those in the *target* coordinate reference system in the reverse transformation. Usage of the same transformation parameters for the reverse transformation will therefore only be valid if the evaluation point coordinates are numerically the same in source and target coordinate reference system and in the same units. That is, $X_{S0} = X_{T0} = X_0$ and $Y_{S0} = Y_{T0} = Y_0$.

Re 3 - The same holds for the scaling factors of the source and target coordinate differences and for the units of measure of the coordinate differences. That is, $m_S = m_T = m$.

Re 4 - If conditions 1, 2 and 3 are all satisfied it then may be possible to use the forward polynomial algorithm with the forward parameters for the reverse transformation. This is the case if the spatial variations in dX and dY around any given location are sufficiently constant. The signs of the polynomial coefficients are then reversed but the scaling factor and the evaluation point coordinates retain their signs. If these spatial variations in dX and dY are too large, for the reverse transformation iteration would be necessary. It is therefore not the algorithm that determines whether a single step solution is sufficient or whether iteration is required, but the desired accuracy combined with the degree of spatial variability of dX and dY.

An example of a reversible polynomial is transformation is ED50 to ED87 (1) for the North Sea. The suitability of this transformation to be described by a reversible polynomial can easily be explained. In the first place both source and target coordinate reference systems are of type geographic 2D. The typical difference in coordinate values between ED50 and ED87 is in the order of 2 metres ($\approx 10^{-6}$ degrees) in the area of application. The polynomial functions are evaluated about central points with coordinates of 55°N, 0° E in both coordinate reference systems. The reduced coordinate differences (*in degrees*) are used as input parameters to the polynomial functions. The output coordinate differences are corrections to the input coordinate offsets of about *10^{-6} degrees*. This difference of several orders of magnitude between input and output values is the property that makes a polynomial function reversible in practice (although not in a formal mathematical sense).

The error made by the polynomial approximation formulas in calculating the reverse correction is of the same order of magnitude as the ratio of output versus input:

output error *output value*

$$\frac{}{\textit{input error}} \approx \frac{}{\textit{input value}} \quad (\approx 10^{-6})$$

As long as the input values (the coordinate offsets from the evaluation point) are orders of magnitude larger than the output (the corrections), and provided the coefficients are used with changed signs, the polynomial transformation may be considered to be reversible.

Hence the EPSG dataset acknowledges two classes of general polynomial functions, reversible and non-reversible, as distinguished by whether or not the coefficients may be used in both forward and reverse transformations, i.e. are reversible. The EPSG dataset does not describe the iterative solution as a separate algorithm. The iterative solution for the reverse transformation, when applicable, is deemed to be implied by the (forward) algorithm.

Example: Reversible polynomial of degree 4 (EPSG dataset coordinate operation method code 9651)
For coordinate transformation ED50 to ED87 (1)

Ordinate 1 of evaluation point: $X_O = \varphi_O$ = 55°00'00.000"N = +55 degrees
Ordinate 2 of evaluation point: $Y_O = \lambda_O$ = 0°00'00.000"E = +0 degrees

Scaling factor for coordinate differences: m = 1.0

Parameters:
A_0 = – 5.56098E-06 A_1 = – 1.55391E-06 ... A_{14} = – 4.01383E-09
B_0 = + 1.48944E-05 B_2 = + 2.68191E-05 ... B_{14} = + 7.62236E-09

Forward calculation for:
Latitude $\varphi_{ED50} = X_s$ = 52°30'30"N = + 52.508333333 degrees
Longitude $\lambda_{ED50} = Y_s$ = 2°E = + 2.0 degrees

$U = m * (X_S - X_0) = m * (\varphi_{ED50} - \varphi_0) = 1.0 * (52.508333333 - 55.0) = - 2.491666667$ degrees
$V = m * (Y_S - Y_0) = m * (\lambda_{ED50} - \lambda_0) = 1.0 * (2.0 - 0.0) = 2.0$ degrees

$dX = (A_0 + A_1U + \ldots + A_{14}V^4) / k_{CD}$
$= [- 5.56098E{-}06 + (- 1.55391E{-}06 * - 2.491666667) + \ldots + (- 4.01383E{-}09 * 2.0\text{^}4)]/1.0$
$= - 3.12958E{-}06$ degrees

$dY = (B_0 + B_1U + \ldots + B_{14}V^4) / k_{CD}$
$= [+1.48944E{-}05 + (2.68191E{-}05 * - 2.491666667) + \ldots + (7.62236E{-}09 * 2.0\text{^}4)]/1.0$
$= +9.80126E{-}06$ degrees

Then: Latitude $\varphi_{ED87} = X_T = X_S + dX$ = 52.508333333 – 3.12958E–06 degrees = 52°30'29.9887"N
Longitude $\lambda_{ED87} = Y_T = Y_S + dY$ = 2°00'00.0353"E

Reverse calculation for coordinate transformation ED50 to ED87 (1).
The transformation method for the ED50 to ED87 (1) coordinate transformation, 4th-order reversible polynomial, is reversible. The same formulas may be applied for the reverse calculation, but coefficients A_0 through A_{14} and B_0 through B_{14} are applied with reversal of their signs. Sign reversal is not applied to the coordinates of the evaluation point or scaling factor for coordinate differences. Thus:
Ordinate 1 of evaluation point: $X_O = \varphi_O$ = 55°00'00.000"N = +55 degrees
Ordinate 2 of evaluation point: $Y_O = \lambda_O$ = 0°00'00.000"E = +0 degrees
Scaling factor for coordinate differences: m = 1.0

A_0 = +5.56098E-06 A_1 = +1.55391E-06 ... A_{14} = +4.01383E-09

B_0 = –1.48944E-05 B_1 = –2.68191E-05 ... B_{14} = –7.62236E-09

Reverse calculation for:

Latitude φ_{ED87} = X_S = 52°30'29.9887"N = +52.5083301944 degrees
Longitude λ_{ED87} = Y_S = 2°00'00.0353"E = +2.0000098055 degrees

U = 1.0 * (52.5083301944 – 55.0) = – 2.4916698056 degrees
V = 1.0 * (2.0000098055 – 0.0) = 2.0000098055 degrees

$$dX = (A_0 + A_1U + ... + A_{14}V^4)/k$$
$$= [+5.56098E\text{-}06 + (1.55391E\text{–}06 * – 2.491666667) + ... $$
$$... + (4.01383E\text{-}09 * 2.0000098055\text{^}4)]/1.0$$
$$= +3.12957E\text{–}06 \text{ degrees}$$

$$dY = (B_0 + B_1.U + ... + B_{14}.V^4)/k$$
$$= [– 1.48944E\text{-}05 + (-2.68191E\text{-}05 * -2.491666667) + ...$$
$$... + (– 7.62236E\text{-}09 * 2.0000098055\text{^}4)]/1.0$$
$$= – 9.80124E\text{–}06 \text{ degrees}$$

Then: Latitude φ_{ED50} = $X_T = X_S + dX$ = 52.5083301944 + 3.12957E–06 degrees = 52°30'30.000"N
Longitude λ_{ED50} = $Y_T = Y_S + dY$ = = 2°00'00.000"E

2.3.1.2 <u>Polynomial transformation with complex numbers</u>

The relationship between two projected coordinate reference systems may be approximated more elegantly by a single polynomial regression formula written in terms of complex numbers. The advantage is that the dependence between the ‘A’ and ‘B’ coefficients (for U and V) is taken into account in the formula, resulting in fewer coefficients for the same order polynomial. A polynomial to degree 3 in complex numbers is used in Belgium. A polynomial to degree 4 in complex numbers is used in The Netherlands for transforming coordinates referenced to the Amersfoort / RD system to and from ED50 / UTM.

$$\begin{aligned} m_T (dX + i\, dY) = {} & (A_1 + i\, A_2) (U + i\, V) + (A_3 + i\, A_4) (U + i\, V)^2 && \text{(to degree 2)} \\ & + (A_5 + i\, A_6) (U + i\, V)^3 && \text{(additional degree 3 terms)} \\ & + (A_7 + i\, A_8) (U + i\, V)^4 && \text{(additional degree 4 terms)} \end{aligned}$$

where $U = m_S (X_S – X_{S0})$
$V = m_S (Y_S – Y_{S0})$
and m_S, m_T are the scaling factors for the coordinate differences in the source and target coordinate reference systems.

The polynomial to degree 4 can alternatively be expressed in matrix form as

$$\begin{pmatrix} m_T.dX \\ \\ m_T.dY \end{pmatrix} = \begin{pmatrix} +A_1 & -A_2 & +A_3 & -A_4 & +A_5 & -A_6 & +A_7 & -A_8 \\ \\ +A_2 & +A_1 & +A_4 & +A_3 & +A_6 & +A_5 & +A_8 & +A_7 \end{pmatrix} * \begin{pmatrix} U \\ V \\ U^2–V^2 \\ 2UV \\ U^3–3UV^2 \\ 3U^2V–V^3 \\ U^4–6U^2V^2 + V^4 \\ 4U^3V–4UV^3 \end{pmatrix}$$

Then as for the general polynomial case above

$X_T = X_S - X_{SO} + X_{TO} + dX$

$Y_T = Y_S - Y_{SO} + Y_{TO} + dY$

where, as above,

X_T, Y_T are coordinates in the target coordinate system,

X_S, Y_S are coordinates in the source coordinate system,

X_{SO}, Y_{SO} are coordinates of the evaluation point in the source coordinate reference system,

X_{TO}, Y_{TO} are coordinates of the evaluation point in the target coordinate reference system.

Note that the zero order coefficients of the general polynomial, A_0 and B_0, have apparently disappeared. In reality they are absorbed by the different coordinates of the source and of the target evaluation point, which in this case, are numerically very different because of the use of two different projected coordinate systems for source and target.

The transformation parameter values (the coefficients) are not reversible. For the reverse transformation a different set of parameter values are required, used within the same formulas as the forward direction.

Example: Complex polynomial of degree 4 (EPSG dataset coordinate operation method code 9653)
Coordinate transformation: Amersfoort / RD New to ED50 / UTM zone 31N (1):

Coordinate transformation parameter name	Formula symbol	Parameter value	Unit
ordinate 1 of the evaluation point in the source CS	X_{SO}	155,000.000	metre
ordinate 2 of the evaluation point in the source CS	Y_{SO}	463,000.000	metre
ordinate 1 of the evaluation point in the target CS	X_{TO}	663,395.607	metre
ordinate 2 of the evaluation point in the target CS	Y_{TO}	5,781,194.380	metre
scaling factor for source CRS coordinate differences	m_S	10^{-5}	
scaling factor for target CRS coordinate differences	m_T	1.0	
A1	A_1	–51.681	coefficient
A2	A_2	+3,290.525	coefficient
A3	A_3	+20.172	coefficient
A4	A_4	+1.133	coefficient
A5	A_5	+2.075	coefficient
A6	A_6	+0.251	coefficient
A7	A_7	+0.075	coefficient
A8	A_8	–0.012	coefficient

For input point:

Easting, $X_{AMERSFOORT/RD} = X_S = 200{,}000.00$ metres

Northing, $Y_{AMERSFOORT/RD} = Y_S = 500{,}000.00$ metres

$U = m_S (X_S - X_{S0}) = (200{,}000 - 155{,}000)\ 10^{-5} = 0.45$

$V = m_S (Y_S - Y_{S0}) = (500{,}000 - 463{,}000)\ 10^{-5} = 0.37$

$dX = (-1{,}240.050) / 1.0$

$dY = (1{,}468.748) / 1.0$

Then: Easting, $\mathbf{E}_{ED50/UTM31} = X_T = X_S - X_{SO} + X_{TO} + dX$

$= 200{,}000 - 155{,}000 + 663{,}395.607 + (-1{,}240.050)$

$= 707{,}155.557$ metres

Northing, $N_{ED50/UTM31N}$ = Y_T = $Y_S - Y_{S0} + Y_{T0} + dY$
= 500,000 – 463,000 + 5,781,194.380 + 1,468.748
= 5,819,663.128 metres

2.3.1.3 Polynomial transformation for Spain

(EPSG dataset coordinate operation method code 9617)

The original geographic coordinate reference system for the Spanish mainland was based on Madrid 1870 datum, Struve 1860 ellipsoid, with longitudes related to the Madrid meridian. Three second-order polynomial expressions have been empirically derived by El Servicio Geográfico del Ejército to transform geographical coordinates based on this system to equivalent values based on the European Datum of 1950 (ED50). The polynomial coefficients derived can be used to transform coordinates from the Madrid 1870 (Madrid) geographic coordinate reference system to the ED50 system. Three pairs of expressions have been derived: each pair is used to calculate the shift in latitude and longitude respectively for (i) a mean for all Spain, (ii) a better fit for the north of Spain, (iii) a better fit for the south of Spain.

The polynomial expressions are:

$$d\varphi \text{ (arc sec)} = A_0 + (A_1 * \varphi_s) + (A_2 * \lambda_s) + (A_3 * H_s)$$
$$d\lambda \text{ (arc sec)} = B_{00} + B_0 + (B_1 * \varphi_s) + (B_2 * \lambda_s) + (B_3 * H_s)$$

where latitude φ_s and longitude λ_s are in decimal degrees referred to the Madrid 1870 (Madrid) geographic coordinate reference system and H_s is gravity-related height in metres. B_{00} is the longitude (in seconds) of the Madrid meridian measured from the Greenwich meridian; it is the value to be applied to a longitude relative to the Madrid meridian to transform it to a longitude relative to the Greenwich meridian.

The results of these expressions are applied through the formulas:

$\varphi_{ED50} = \varphi_{M1870(M)} + d\varphi$

and $\lambda_{ED50} = \lambda_{M1870(M)} + d\lambda$.

Example:

Input point coordinate reference system: Madrid 1870 (Madrid) (geographic 2D)

Latitude φ_s = 42°38'52.77"N
= +42.647992 degrees

Longitude λ_s = 3°39'34.57"E of Madrid
= +3.659603 degrees from the Madrid meridian.

Gravity-related height H_s = 0 m

For the north zone transformation:

A_0 = 11.328779	B_{00} = -13276.58
A_1 = -0.1674	B_0 = 2.5079425
A_2 = -0.03852	B_1 = 0.8352
A_3 = 0.0000379	B_2 = -0.00864
	B_3 = -0.0000038

$d\varphi$ = +4.05 seconds

Then latitude φ_{ED50} = 42°38'52.77"N + 4.05"
= 42°38'56.82"N

$d\lambda$ = -13238.484seconds = -3°40'38.484"

Then longitude λ_{ED50} = 3°39'34.57"E – 3°40'38.484"

= 0°01'03.914"W of Greenwich.

2.3.2 Miscellaneous Linear Coordinate Operations

An affine 2D transformation is used for converting or transforming a coordinate reference system possibly with non-orthogonal axes and possibly different units along the two axes to an isometric coordinate reference system (i.e. a system of which the axes are orthogonal and have equal scale units, for example a projected CRS). The transformation therefore involves a change of origin, differential change of axis orientation and a differential scale change. The EPSG dataset distinguishes four methods to implement this class of coordinate operation:

1) the parametric representation,
2) the geometric representation,
3) a simplified case of the geometric representation known as the Similarity Transformation in which the degrees of freedom are constrained.
4) a variation of the geometric representation for seismic bin grids.

2.3.2.1 Affine Parametric Transformation

(EPSG dataset coordinate operation method code 9624)

Mathematical and survey literature usually provides the parametric representation of the affine transformation. The parametric algorithm is commonly used for rectification of digitised maps. It is often embedded in CAD software and Geographical Information Systems where it is frequently referred to as "rubber sheeting".

The formula in matrix form is as follows:

$$\boldsymbol{V_T} = \boldsymbol{V_{TO}} + \boldsymbol{R} * \boldsymbol{V_S}$$

where:

$$\boldsymbol{V_T} = \begin{pmatrix} X_T \\ Y_T \end{pmatrix} \quad \boldsymbol{V_{TO}} = \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} \quad \boldsymbol{R} = \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{V_S} = \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

or using algebraic coefficients:

$$X_T = A_0 + A_1 * X_S + A_2 * Y_S$$
$$Y_T = B_0 + B_1 * X_S + B_2 * Y_S$$

where

X_T, Y_T are the coordinates of a point P in the target coordinate reference system;
X_S, Y_S are the coordinates of P in the source coordinate reference system.

This form of describing an affine transformation is analogous to the general polynomial transformation formulas (section 3.1 above). Although it is somewhat artificial, an affine transformation could be considered to be a first order general polynomial transformation but without the reduction to source and target evaluation points.

Reversibility

The reverse operation is another affine parametric transformation using the same formulas but with different parameter values. The reverse parameter values, indicated by a prime ('), can be calculated from those of the forward operation as follows:

$D \;\; = A_1 * B_2 \; - \; A_2 * B_1$
$A_0' = (A_2 * B_0 \; - \; B_2 * A_0) / D$
$B_0' = (B_1 * A_0 \; - \; A_1 * B_0) / D$
$A_1' = +B_2 / D$
$A_2' = - A_2 / D$
$B_1' = - B_1 / D$
$B_2' = +A_1 / D$

Then

$$X_S = A_0' + A_1' * X_T + A_2' * Y_T$$
$$Y_S = B_0' + B_1' * X_T + B_2' * Y_T$$

Or in matrix form:

$$\boldsymbol{V_S} = \boldsymbol{R}^{-1} * (\boldsymbol{V_T} - \boldsymbol{V_{TO}})$$

2.3.2.2 Affine General Geometric Transformation

(EPSG dataset coordinate operation method code 9623)

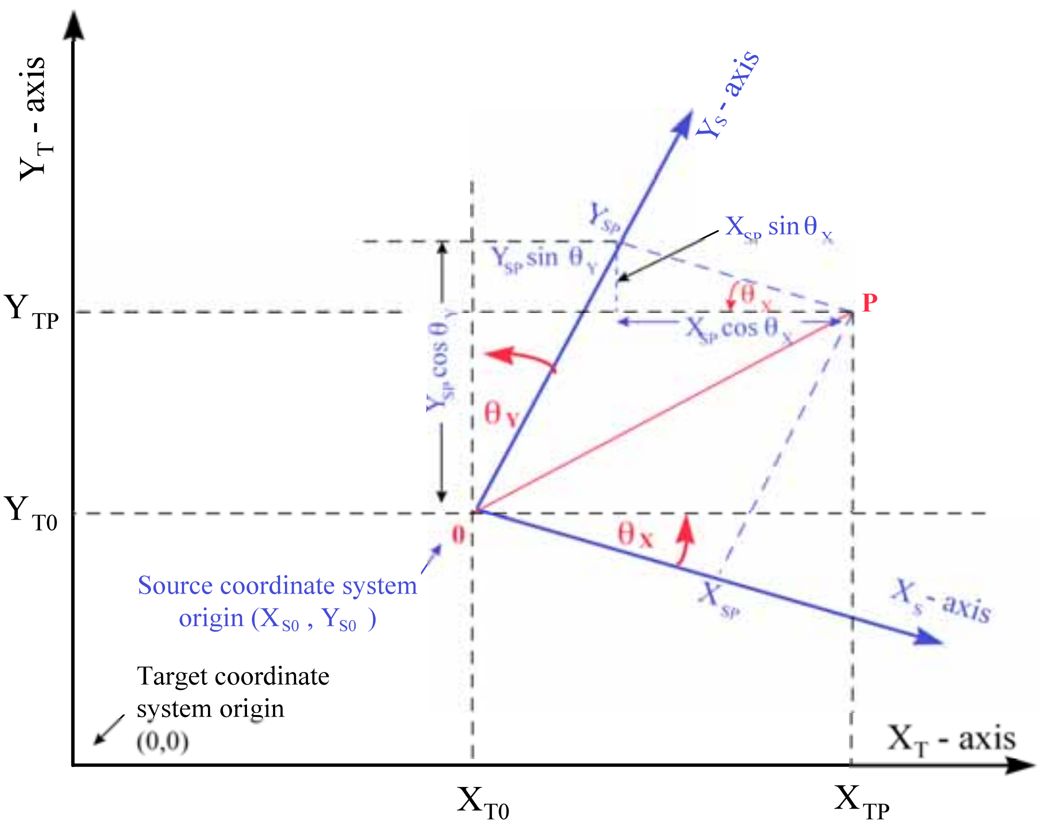

Figure 12. Geometric representation of the affine coordinate transformation
(Please note that to prevent cluttering of the figure the scale parameters of the X_s and Y_s axes have been omitted).

From the diagram above it can be seen that:

$$X_{TP} = X_{TO} + Y_{SP} * \sin\theta_Y + X_{SP} * \cos\theta_X = X_{TO} + X_{SP} * \cos\theta_X + Y_{SP} * \sin\theta_Y$$
$$Y_{TP} = Y_{TO} + Y_{SP} * \cos\theta_Y - X_{SP} * \sin\theta_X = Y_{TO} - X_{SP} * \sin\theta_X + Y_{SP} * \cos\theta_Y$$

The scaling of both source and target coordinate reference systems adds some complexity to this formula. The operation will often be applied to transform an engineering coordinate reference system to a projected coordinate reference system. The orthogonal axes of the projected coordinate reference system have identical same units. The engineering coordinate reference system may have different units of measure on its two axes: these have scale ratios of M_X and M_Y respective to the axes of the projected coordinate reference system.

The projected coordinate reference system is nominally defined to be in well-known units, e.g. metres. However, the distortion characteristics of the map projection only preserve true scale along certain defined lines or curves, hence the projected coordinate reference system's unit of measure is strictly speaking only valid along those lines or curves. Everywhere else its scale is distorted by the map projection. For conformal map projections the distortion at any point can be expressed by the point scale factor 'k' for that point. Please note that this point scale factor 'k' should NOT be confused with the scale factor at the natural origin of the projection, denominated by 'k_0'. (For non-conformal map projections the scale distortion at a point is bearing-dependent and will not be described in this document).

It has developed as working practice to choose the origin of the source (engineering) coordinate reference system as the point in which to calculate this point scale factor 'k', although for engineering coordinate reference systems with a large coverage area a point in the middle of the area may be a better choice.

Adding the scaling between each pair of axes and dropping the suffix for point P, after rearranging the terms we have the geometric representation of the affine transformation:

$$X_T = X_{TO} + X_S * k * M_X * \cos\theta_X + Y_S * k * M_Y * \sin\theta_Y$$
$$Y_T = Y_{TO} - X_S * k * M_X * \sin\theta_X + Y_S * k * M_Y * \cos\theta_Y$$

where:

X_{TO}, Y_{TO} = the coordinates of the origin point of the source coordinate reference system, expressed in the target coordinate reference system;
M_X, M_Y = the length of one unit of the source axis, expressed in units of the target axis, for the first and second source and target axes pairs respectively;
k = point scale factor of the target coordinate reference system at a chosen reference point;
θ_X, θ_Y = the angles about which the source coordinate reference system axes X_S and Y_S must be rotated to coincide with the target coordinate reference system axes X_T and Y_T respectively (counter-clockwise being positive).

Alternatively, in matrix form:

$$\boldsymbol{V_T} = \boldsymbol{V_{TO}} + \boldsymbol{R_1} * k * \boldsymbol{S_1} * \boldsymbol{V_S}$$

where:

$$\boldsymbol{V_T} = \begin{pmatrix} X_T \\ Y_T \end{pmatrix} \qquad \boldsymbol{V_{TO}} = \begin{pmatrix} X_{TO} \\ Y_{TO} \end{pmatrix} \qquad \boldsymbol{V_S} = \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

and

$$\boldsymbol{R_1} = \begin{pmatrix} \cos\theta_X & \sin\theta_Y \\ -\sin\theta_X & \cos\theta_Y \end{pmatrix} \qquad \boldsymbol{S_1} = \begin{pmatrix} M_X & 0 \\ 0 & M_Y \end{pmatrix}$$

or

$$\begin{pmatrix} X_T \\ Y_T \end{pmatrix} = \begin{pmatrix} X_{TO} \\ Y_{TO} \end{pmatrix} + \begin{pmatrix} \cos\theta_X & \sin\theta_Y \\ -\sin\theta_X & \cos\theta_Y \end{pmatrix} * k * \begin{pmatrix} M_X & 0 \\ 0 & M_Y \end{pmatrix} * \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

Comparing the algebraic representation with the parameters of the parameteric form in section 2.3.2.1 above it can be seen that the parametric and geometric forms of the affine coordinate transformation are related as follows:

$A_0 = X_{TO}$
$A_1 = k * M_X * \cos \theta_X$
$A_2 = k * M_Y * \sin \theta_Y$
$B_0 = Y_{TO}$
$B_1 = -k * M_X * \sin \theta_X$
$B_2 = k * M_Y * \cos \theta_Y$

Reversibility
For the Affine Geometric Transformation, the reverse operation can be described by a different formula, as shown below, in which the same parameter values as the forward transformation may be used. In matrix form:

$$\boldsymbol{V_S} = (1/k) * \boldsymbol{S_1}^{-1} * \boldsymbol{R}_1^{-1} * (\boldsymbol{V_T} - \boldsymbol{V_{TO}})$$

or

$$\begin{pmatrix} X_S \\ Y_S \end{pmatrix} = \frac{1}{k\,.Z} * \begin{pmatrix} 1/M_X & 0 \\ 0 & 1/M_Y \end{pmatrix} * \begin{pmatrix} \cos \theta_Y & -\sin \theta_Y \\ \sin \theta_X & \cos \theta_X \end{pmatrix} * \begin{pmatrix} X_T - X_{TO} \\ Y_T - Y_{TO} \end{pmatrix}$$

where $Z = \cos(\theta_X - \theta_Y)$;

Algebraically:
$X_S = [(X_T - X_{TO}) * \cos \theta_Y - (Y_T - Y_{TO}) * \sin \theta_Y] / [k * M_X * \cos(\theta_X - \theta_Y)]$
$Y_S = [(X_T - X_{TO}) * \sin \theta_X + (Y_T - Y_{TO}) * \cos \theta_X] / [k * M_Y * \cos(\theta_X - \theta_Y)]$

Orthogonal case
If the source coordinate reference system happens to have orthogonal axes, that is both axes are rotated through the same angle to bring them into the direction of the orthogonal target coordinate reference system axes, i.e. $\theta_X = \theta_Y = \theta$, then the Affine Geometric Transformation can be simplified. In matrix form this is:

$$\boldsymbol{V_T} = \boldsymbol{V_{TO}} + \boldsymbol{R_2} * k * \boldsymbol{S_1} * \boldsymbol{V_S}$$

where $\boldsymbol{V_T}$, $\boldsymbol{V_{TO}}$, $\boldsymbol{S_1}$ and $\boldsymbol{V_S}$ are as in the general case but

$$\boldsymbol{R_2} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

Alternatively,

$$\begin{pmatrix} X_T \\ Y_T \end{pmatrix} = \begin{pmatrix} X_{TO} \\ Y_{TO} \end{pmatrix} + \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} * k * \begin{pmatrix} M_X & 0 \\ 0 & M_Y \end{pmatrix} * \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

Algebraically:
$X_T = X_{TO} + X_S * k * M_X * \cos \theta + Y_S * k * M_Y * \sin \theta$
$Y_T = Y_{TO} - X_S * k * M_X * \sin \theta + Y_S * k * M_Y * \cos \theta$

where:

X_{TO}, Y_{TO} = the coordinates of the origin point of the source coordinate reference system, expressed in the target coordinate reference system;

M_X, M_Y = the length of one unit of the source axis, expressed in units of the target axis, for the X axes and the Y axes respectively;
k = the point scale factor of the target coordinate reference system at a chosen reference point;
θ = the angle through which the source coordinate reference system axes must be rotated to coincide with the target coordinate reference system axes (counter-clockwise is positive). Alternatively, the bearing (clockwise positive) of the source coordinate reference system Y_S-axis measured relative to target coordinate reference system north.

The reverse formulas of the general case can also be simplified by replacing θ_X and θ_Y with θ. In matrix form:

$$\boldsymbol{V_S} = (1/k) * \boldsymbol{S_1}^{-1} * \boldsymbol{R_2}^{-1} * (\boldsymbol{V_T} - \boldsymbol{V_{TO}})$$

or

$$\begin{pmatrix} X_S \\ Y_S \end{pmatrix} = \frac{1}{k} * \begin{pmatrix} 1/M_X & 0 \\ 0 & 1/M_Y \end{pmatrix} * \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} * \begin{pmatrix} X_T - X_{TO} \\ Y_T - Y_{TO} \end{pmatrix}$$

Algebraically:

$$X_S = [(X_T - X_{TO}) * \cos\theta - (Y_T - Y_{TO}) * \sin\theta] / [k * M_X]$$
$$Y_S = [(X_T - X_{TO}) * \sin\theta + (Y_T - Y_{TO}) * \cos\theta] / [k * M_Y]$$

In the EPSG dataset this orthogonal case has been deprecated. The formulas for the general case should be used, inserting θ for both θ_X and θ_Y. The case has been documented as part of the progression through increasing constraints on the degrees of freedom between the general case and the Similarity Transformation.

2.3.2.3 <u>Similarity Transformation</u>

(EPSG dataset coordinate operation method code 9621)

If the source coordinate reference system has orthogonal axes and also happens to have axes of the same scale, that is both axes are scaled by the same factor to bring them into the scale of the target coordinate reference system axes (i.e. $M_X = M_Y = M$), then the orthogonal case of the Affine Geometric Transformation can be simplified further to a Similarity Transformation.

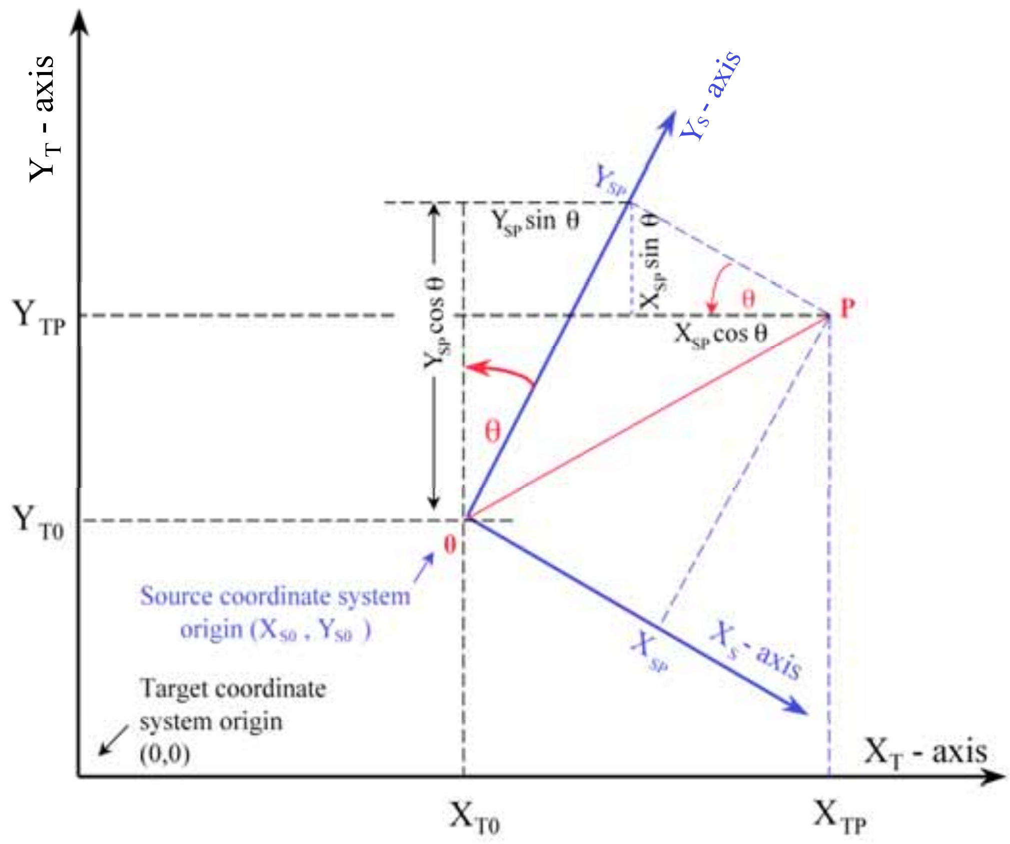

Figure 13. Similarity Transformation

From the above diagram the Similarity Transformation in algebraic form is:

$$X_{TP} = X_{TO} + Y_{SP} * M * \sin\theta \ + X_{SP} * M * \cos\theta$$
$$Y_{TP} = Y_{TO} + Y_{SP} * M * \cos\theta \ - \ X_{SP} * M * \sin\theta$$

Dropping the suffix for point P and rearranging the terms

$$X_T = X_{TO} \ + X_S * M * \cos\theta \ + Y_S * M * \sin\theta$$
$$Y_T = Y_{TO} \ - \ X_S * M * \sin\theta \ + Y_S * M * \cos\theta$$

where:

X_{TO} , Y_{TO} = the coordinates of the origin point of the source coordinate reference system expressed in the target coordinate reference system;

M = the length of one unit in the source coordinate reference system expressed in units of the target coordinate reference system;

θ = the angle about which the axes of the source coordinate reference system need to be rotated to coincide with the axes of the target coordinate reference system, counter-clockwise being positive. Alternatively, the bearing of the source coordinate reference system Y_S-axis measured relative to target coordinate reference system north.

The Similarity Transformation can also be described as a special case of the Affine Parametric Transformation where coefficients $A_1 = B_2$ and $A_2 = - B_1$.

In matrix form:

$$\boldsymbol{V_T} = \boldsymbol{V_{TO}} + M * \boldsymbol{R_2} * \boldsymbol{V_S}$$

where $\boldsymbol{V_T}$, $\boldsymbol{V_{TO}}$, $\boldsymbol{R_2}$ and $\boldsymbol{V_S}$ are as in the Affine Orthogonal Geometric Transformation method, or

$$\begin{pmatrix} X_T \\ Y_T \end{pmatrix} = \begin{pmatrix} X_{TO} \\ Y_{TO} \end{pmatrix} + M * \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} * \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

<u>Reversibility</u>

The reverse formula for the Similarity Transformation, in matrix form, is:

$$\boldsymbol{V_S} = (1/M) * \boldsymbol{R_2}^{-1} * (\boldsymbol{V_T} - \boldsymbol{V_{TO}})$$

or

$$\begin{pmatrix} X_S \\ Y_S \end{pmatrix} = \frac{1}{M} * \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} * \begin{pmatrix} X_T - X_{TO} \\ Y_T - Y_{TO} \end{pmatrix}$$

Algebraically:

$X_S = [(X_T - X_{TO}) * \cos\theta \;-\; (Y_T - Y_{TO}) * \sin\theta\,] / [M\,]$
$Y_S = [(X_T - X_{TO}) * \sin\theta \;+\; (Y_T - Y_{TO}) * \cos\theta] / [M\,]$

Example
Tombak LNG Plant Grid to Nakhl-e Ghanem / UTM zone 39N

Parameters of the Similarity Transformation:
X_{TO} = 611267.2865 metres
Y_{TO} = 3046565.8255 metres
M = 0.9997728332
θ = 315 degrees

Forward computation for plant grid coordinates x (= X_S) = 20000m, y (= Y_S) = 10000m:

X_T = UTM E = 611267.2865 + 14138.9230 + (–7069.4615)
= 618336.748 m

Y_T = UTM N = 3046565.8255 – (–14138.9230) + 7069.4615
= 3067774.210 m

Reverse computation for UTM coordinates 618336.748 m E, 3067774.210 m:

Plant x = [4998.8642 – (–14996.5925)] / 0.9997728332
= 20000.000 m

Plant y = [(– 4998.8642) + 14996.5925)] / 0.9997728332
= 10000.000 m

When to use the Similarity Transformation

Similarity Transformations can be used when source and target coordinate reference systems
- each have orthogonal axes,
- each have the same scale along both axes,

and

- both have the same units of measure,

for example between engineering plant grids and projected coordinate reference systems.

Coordinate Operations between two coordinate reference systems where in either system either the scale along the axes differ or the axes are not orthogonal should be defined as an Affine Transformation in either the parametric or geometric form. But for seismic bin grids see the following section.

2.3.2.4 UKOOA P6 Seismic Bin Grid Transformation
(EPSG dataset coordinate operation method code 9666)

The UKOOA P6/98 exchange format describes a special case of the Affine Geometric Transformation in which
- the source coordinate reference system is a grid;
- its axes are orthogonal;

and one or both of the following may apply:
- the origin of the bin grid (source coordinate reference system) may be assigned non-zero bin grid coordinates;
- the bin grid (source coordinate reference system) units may increase in increments other than 1, i.e. Inc_{SX} and Inc_{SY}

The method is also described in the SEG-Y revision 1 seismic data exchange format.

The defining parameters are:

UKOOA P6 term	**Equivalent EPSG dataset term**
Bin grid origin (Io)	Ordinate 1 of evaluation point in source CRS (X_{SO})
Bin grid origin (Jo)	Ordinate 2 of evaluation point in source CRS (Y_{SO})
Map grid easting of bin grid origin (Eo)	Ordinate 1 of evaluation point in target CRS (X_{TO})
Map grid northing of bin grid origin (No)	Ordinate 2 of evaluation point in target CRS (Y_{TO})
Scale factor of bin grid (SF)	Point scale factor (k)
Nominal bin width along I axis (I_bin_width)	Scale factor for source coordinate reference system first axis (M_X)
Nominal bin width along J axis (J_bin_width)	Scale factor for source coordinate reference system second axis (M_y)
Grid bearing of bin grid J axis (θ)	Rotation angle of source coordinate reference system axes (θ)
Bin node increment on I axis (I_bin_inc)	Bin node increment on I-axis
Bin node increment on J axis (J_bin_inc)	Bin node increment on J-axis

In the orthogonal case of the Affine Geometric Transformation formulas, the terms X_S, Y_S, M_X and M_Y are replaced by $(X_S - X_{SO})$, $(Y_S - Y_{SO})$, (M_X / Inc_{SX}) and (M_Y / Inc_{SY}) respectively. Thus the forward transformation from bin grid to map grid (source to target coordinate reference system) is:

$$\boldsymbol{V_T} = \boldsymbol{V_{TO}} + \boldsymbol{R_2} * k * \boldsymbol{S_2} * \boldsymbol{V_2}$$

where, as in the orthogonal case of the Affine Geometric Transformation method:

$$\boldsymbol{V_T} = \begin{pmatrix} X_T \\ Y_T \end{pmatrix} \qquad \boldsymbol{V_{TO}} = \begin{pmatrix} X_{TO} \\ Y_{TO} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{R_2} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

but where

$$\boldsymbol{S_2} = \begin{pmatrix} M_X / Inc_{SX} & 0 \\ 0 & M_Y / Inc_{SY} \end{pmatrix} \quad \text{and} \quad \boldsymbol{V_2} = \begin{pmatrix} X_S - X_{SO} \\ Y_S - Y_{SO} \end{pmatrix}$$

That is,

$$\begin{pmatrix} X_T \\ Y_T \end{pmatrix} = \begin{pmatrix} X_{TO} \\ Y_{TO} \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} * k * \begin{pmatrix} M_X / Inc_{SX} & 0 \\ 0 & M_Y / Inc_{SY} \end{pmatrix} * \begin{pmatrix} X_S - X_{SO} \\ Y_S - Y_{SO} \end{pmatrix}$$

Algebraically:
$X_T = X_{TO} + [(X_S - X_{SO}) * \cos\theta * k * M_X / Inc_{SX}] + [(Y_S - Y_{SO}) * \sin\theta * k * M_Y / Inc_{SY}]$
$Y_T = Y_{TO} - [(X_S - X_{SO}) * \sin\theta * k * M_X / Inc_{SX}] + [(Y_S - Y_{SO}) * \cos\theta * k * M_Y / Inc_{SY}]$

Using the symbol notation in the UKOOA P6/98 document these expressions are:

$$\begin{pmatrix} E \\ N \end{pmatrix} = \begin{pmatrix} E_O \\ N_O \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} * SF * \begin{pmatrix} I\_bin\_width / I\_bin\_inc & 0 \\ 0 & J\_bin\_width / J\_bin\_inc \end{pmatrix} * \begin{pmatrix} I - I_O \\ J - J_O \end{pmatrix}$$

and

$E = E_O + [(I - I_O) * \cos\theta * SF * I\_bin\_width / I\_bin\_inc]$
$\quad + [(J - J_O) * \sin\theta * SF * J\_bin\_width / J\_bin\_inc]$

$N = N_0 - [(I - I_O) * \sin\theta * SF * I\_bin\_width / I\_bin\_inc]$
$\quad + [(J - J_O) * \cos\theta * SF * J\_bin\_width / J\_bin\_inc]$

For the reverse transformation (map grid to bin grid):

$$\boldsymbol{V_S} = (1/k) * \boldsymbol{S_2}^{-1} * \boldsymbol{R_2}^{-1} * (\boldsymbol{V_T} - \boldsymbol{V_{TO}}) + \boldsymbol{V_{SO}}$$

or

$$\begin{pmatrix} X_S \\ Y_S \end{pmatrix} = 1/k * \begin{pmatrix} Inc_{SX} / M_X & 0 \\ 0 & Inc_{SY} / M_Y \end{pmatrix} * \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin & \cos\theta \end{pmatrix} * \begin{pmatrix} X_T - X_{TO} \\ Y_T - Y_{TO} \end{pmatrix} + \begin{pmatrix} X_{SO} \\ Y_{SO} \end{pmatrix}$$

or algebraically:
$X_S = \{[(X_T - X_{TO}) * \cos\theta - (Y_T - Y_{TO}) * \sin\theta] * [Inc_{SX} / (k * M_X)]\} + X_{SO}$
$Y_S = \{[(X_T - X_{TO}) * \sin\theta + (Y_T - Y_{TO}) * \cos\theta] * [Inc_{SY} / (k * M_Y)]\} + Y_{SO}$

Using the symbol notation in the UKOOA P6/98 document these reverse expressions are:

$$\begin{pmatrix} I \\ J \end{pmatrix} = 1/SF * \begin{pmatrix} I\_bin\_inc / I\_bin\_width & 0 \\ 0 & J\_bin\_inc / I\_bin\_width \end{pmatrix} * \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin & \cos\theta \end{pmatrix} * \begin{pmatrix} E - E_O \\ N - N_O \end{pmatrix} + \begin{pmatrix} I_O \\ J_O \end{pmatrix}$$

and

$I = \{[(E - E_O) * \cos\theta \ - \ (N - N_O) * \sin\theta\ ] * [I\_bin\_inc \ / \ (SF * I\_bin\_width)]\} + I_O$
$J = \{[(E - E_O) * \sin\theta \ + \ (N - N_O) * \cos\theta] * [J\_bin\_inc \ / \ (SF * J\_bin\_width)]\} + J_O$

Example:
This example is given in the UKOOA P6/98 document. Source coordinate reference system: imaginary 3D seismic acquisition bin grid. The two axes are orthogonal, but the bin width on the I-axis (X_S axis) is 25 metres, whilst the bin width on the J-axis (Y_S axis) is 12.5 metres. The origin of the grid has bin values of 1,1.

The target coordinate reference system is a projected CRS (WGS 84 / UTM Zone 31N) upon which the origin of the bin grid is defined at E = 456781.0, N = 5836723.0. The projected coordinate reference system point scale factor at the bin grid origin is 0.99984.

In the map grid (target CRS), the bearing of the bin grid (source CRS) I and J axes are 110° and 20° respectively. Thus the angle through which the bin grid axes need to be rotated to coincide with the map grid axes is +20 degrees.

The transformation parameter values are:

Parameter	EPSG symbol	P6 symbol	Parameter value
Bin grid origin I	X_{SO}	Io	1
Bin grid origin J	Y_{SO}	Jo	1
Bin grid origin Easting	X_{TO}	Eo	456781.00 m
Bin grid origin Northing	Y_{TO}	No	5836723.00 m
Scale factor of bin grid	k	SF	0.99984
Bin Width on I-axis	M_X	I_bin_width	25 m
Bin Width on J-axis	M_Y	J_bin_width	12.5 m
Map grid bearing of bin grid J-axis	θ	θ	20 deg
Bin node increment on I-axis	Inc_{SX}	I_bin_inc	1
Bin node increment on J-axis	Inc_{SY}	J_bin_inc	1

Forward calculation for centre of bin with coordinates: I = 300, J = 247:
X_T = Easting $= X_{TO} \ + \ [(X_S - X_{SO}) * \cos\theta * k * M_X / Inc_{SX}] \ + \ [(Y_S - Y_{SO}) * \sin\theta * k * M_Y / Inc_{SY}]$
= 456781.000 + 7023.078 + 1051.544
= 464855.62 m.

Y_T = Northing $= Y_{TO} \ - \ [(X_S - X_{SO}) * \sin\theta * k * M_X / Inc_{SX}] \ + \ [(Y_S - Y_{SO}) * \cos\theta * k * M_Y / Inc_{SY}]$
= 5836723.000 - 2556.192 + 2889.092
= 5837055.90 m.

Reverse calculation for this point 464 855.62mE, 5 837 055.90mN:
$X_S = \{[(X_T - X_{TO}) * \cos\theta \ - \ (Y_T - Y_{TO}) * \sin\theta\ ] * [Inc_{SX} \ / \ (k * M_X)]\} + X_{SO}$
= 300 bins,

$Y_S = \{[(X_T - X_{TO}) * \sin\theta \ + \ (Y_T - Y_{TO}) * \cos\theta] * [Inc_{SY} \ / \ (k * M_Y)]\} + Y_{SO}$
= 247 bins

2.4 Coordinate Transformations

2.4.1 Offsets - general

Several transformation methods which utilise offsets in coordinate values are recognised. The offset methods may be in n-dimensions. These include longitude rotations, geographical coordinate offsets and vertical offsets.

Mathematically, if the origin of a one-dimensional coordinate system is shifted along the positive axis and placed at a point with ordinate A, then the transformation formula is:

$$X_{new} = X_{old} - A$$

However it is common practice in coordinate system transformations to apply the shift as an addition, with the sign of the shift parameter value having been suitably reversed to compensate for the practice. Since 1999 this practice has been adopted for the EPSG dataset. Hence transformations allow calculation of coordinates in the target system by adding a correction parameter to the coordinate values of the point in the source system:

$$X_t = X_s + A$$

where X_s and X_t are the values of the coordinates in the source and target coordinate systems and A is the value of the transformation parameter to transform source coordinate reference system coordinate to target coordinate reference system coordinate.

Offset methods are reversible. For the reverse transformation, the offset parameter value is applied with its sign reversed.

2.4.2 Transformations between Vertical Coordinate Reference Systems

2.4.2.1 Vertical Offset

(EPSG dataset coordinate operation method code 9616)

As described in 2.4.1, a vertical offset allows calculation of coordinates in the target vertical coordinate reference system by adding a correction parameter A to the coordinate values of the point in the source system:

$$X_2 = X_1 + A_{1>2}$$

where

X_2 = value in the forward target vertical coordinate reference system.

X_1 = value in the forward source vertical coordinate reference system.

$A_{1>2}$ is the offset to be applied for the transformation from CRS 1 to CRS 2. Its value for the forward calculation is the value of the origin of the source CRS 1 in the target CRS 2.

For the **reverse** transformation from CRS 2 to CRS 1 the same formula is used but with the sign of the offset $A_{1>2}$ reversed:

$$X_1 = X_2 + (-A_{1>2})$$

Change of axis direction

The above formulas apply only when the positive direction of the axis of each CRS is the same. If there is a requirement to transform heights in the source CRS to depths in the target CRS or to transform depths in the source CRS to heights in the target CRS, the formulas must be modified to:

for the forward transformation: $X_2 = mX_1 + A_{1>2}$

for the reverse transformation: $X_1 = m[X_2 + (-A_{1>2})]$

where m is a direction modifier,

m = +1 if the transformation involves no change of axis direction, i.e. height to height or depth to depth

m = –1 if the transformation involves a change of axis direction, i.e. height to depth or depth to height

These modified formulas remain valid whether or not there is a change in axis direction.

Change of unit

A further modification allows for source CRS axis, target CRS axis or offset to be in different units giving the general formulas:

for the forward transformation: $X_2 = \{m * (X_1 * U_1) + (A_{1>2} * U_A)\} / U_2$

for the reverse transformation: $X_1 = \{m * [(X_2 * U_2) + (-A_{1>2} * U_A)]\} / U_1$

where U_1 U_2 and U_A are unit conversion ratios for the two systems and the offset value respectively. U = [(factor b) / (factor c)] from the EPSG Dataset Unit of Measure table, populated with respect to the linear base unit, metre. U has a value of 0.3048 for the international foot.

Example:

For coordinate transformation: KOC CD height to KOC WD depth (ft) (1), code 5453:

Transformation Parameter: Vertical Offset $A_{1>2}$ = 15.55 ft

Source CRS axis direction is 'up' and Target CRS axis direction is 'down', hence $m = -1$

Offset unit = "foot" for which (from UoM table) b = 0.3048 and c = 1, then $U_A = b/c = 0.3048$

Source CRS (KOC CD height) coordinate axis unit = "metre", b = 1, c = 1, then $U_s = 1$

Target CRS (KOC WD depth) coordinate axis unit = "foot", b = 0.3048, c = 1, then $U_t = 0.3048$

Consider a point having a gravity-related height H_{CD} in the KOC Construction Datum height system of 2.55 m. Its value in the KOC Well Datum depth (ft) system is

$$D_{WD} = \{ -1 * (2.55 * 1) + (15.55 * 3048)\} / 0.3048$$
$$= 7.18 \text{ ft}$$

For the reverse calculation to transform the Well Datum depth of 7.18 ft to Construction Datum height:

$$H_{CD} = \{ -1 * [(7.18 * 0.3048) + (-(15.55) * 0.3048)]\} / 1$$
$$= 2.55 \text{ m}$$

2.4.2.2 Vertical Offset by Interpolation of Gridded Data

The relationship between some gravity-related coordinate reference systems is available through gridded data sets of offsets (sometimes called height differences). The vertical offset at a point is first interpolated within the grid of values.

For the purposes of interpolation, horizontal coordinates of the point are required. However the transformation remains 1-dimensional. Although the providers of some gridded data sets suggest a particular interpolation method within the grid, generally the density of grid nodes should be such that any reasonable grid interpolation method will give the same offset value within an appropriately small tolerance. Bi-linear interpolation is the most usual grid interpolation mechanism. The EPSG dataset differentiates methods by the format of the gridded data file. The grid file format is given in documentation available from the information source. An example is *Vertcon* (EPSG dataset coordinate operation method code 9658) which is used by the US National Geodetic Survey for transformation between the NGVD29 and NAVD88 gravity-related height systems. Because the difference in NAD27 and NAD83 horizontal coordinate values of a point is insignificant in comparison to the rate of change of height offset, interpolation within the Vertcon gridded data file may be made in either NAD27 or NAD83 horizontal systems.

Once the vertical offset value has been derived from the grid it is applied through the formulas given in the previous section.

2.4.2.3 Vertical Offset and Slope

(EPSG dataset coordinate operation method code 9657)

In Europe, national vertical systems are related to the pan-European vertical system through three transformation parameters and the formula:

$X_2 = m * X_1 + \{A_{1>2} + [I_{\varphi 1>2} * \rho_O * (\varphi - \varphi_O)] + [I_{\lambda 1>2} * \nu_O * (\lambda - \lambda_O) * \cos\varphi]\}$ where

X_2 = value in the target vertical coordinate reference system.
X_1 = value in the source vertical coordinate reference system.
m indicates a direction change of the CRS axis:
$m = +1$ when no direction change takes place (height to height or depth to depth),
$m = -1$ in case of a direction change (height to depth or depth to height).

$A_{1>2}$ is the offset to be applied for the transformation from CRS 1 to CRS 2. Its value is the value of the origin of the source CRS 1 in the target CRS 2.

$I_{\varphi 1>2}$ is the value in radians of the slope parameter in the latitude domain, i.e. in the plane of the meridian, derived at an evaluation point with coordinates of φ_O , λ_O. When I_φ is positive then to the north of the evaluation point latitude φ_O the source and target CRS surfaces converge.

$I_{\lambda 1>2}$ is the value in radians of the slope parameter in the longitude domain, i.e. perpendicular to the plane of the meridian. When I_λ is positive then to the east of the evaluation point longitude λ_O the CRS surfaces converge.

ρ_O is the radius of curvature of the meridian at latitude φ_O,
where $\rho_O = a(1 - e^2)/(1 - e^2\sin^2\varphi_O)^{3/2}$

ν_O is the radius of curvature on the prime vertical (i.e. perpendicular to the meridian) at latitude φ_O,
where $\nu_O = a /(1 - e^2\sin^2\varphi_O)^{1/2}$

φ , λ are the horizontal coordinates of the point in the ETRS89 coordinate reference system, in radians.

φ_O , λ_O are the coordinates of the evaluation point in the ETRS89 coordinate reference system, in radians.

The horizontal location of the point must always be given in ETRS89 terms. Care is required where compound coordinate reference systems are in use: if the horizontal coordinates of the point are known in the local CRS they must first be transformed to ETRS89 values.

Reversibility
Similarly to the Vertical Offset method described in previous sections above, the Vertical Offset and Slope method is reversible using a slightly different formula to the forward formula and in which the signs of the parameters A, I_φ and I_λ from the forward transformation are reversed in the reverse transformation:

$$X_1 = m * \{X_2 + –A_{1>2} + [–I_{\varphi 1>2} * \rho_O * (\varphi – \varphi_O)] + [–I_{\lambda 1>2} * \nu_O * (\lambda – \lambda_O) * \cos\varphi]\}$$

Example:
For coordinate transformation LN02 height to EVRF2000 height (1)

Ordinate 1 of evaluation point:	φ_{S0} =	46°55'N	=	0.818850307 rad
Ordinate 2 of evaluation point:	λ_{S0} =	8°11'E (of Greenwich)	=	0.142826110 rad
Transformation Parameters:	A =	–0.245 m		
	I_φ =	–0.210"	=	-0.000001018 rad
	I_λ =	–0.032"	=	-0.000000155 rad

Source axis direction is "up", target axis direction is "up", m = +1

Consider a point having a gravity-related height in the LN02 system (H_s) of 473.0m and with horizontal coordinates in the ETRS89 geographical coordinate reference system of:

Latitude φ_{ETRS89} = 47°20'00.00"N = 0.826122513 rad
Longitude λ_{ETRS89} = 9°40'00.00"E = 0.168715161 rad

ETRS89 uses the GRS1980 ellipsoid for which a = 6378137 m and 1/f = 298.25722221

Then
ρ_O = 6369526.88 m
I_φ term = –0.047 m
ν_O = 6389555.64 m
I_λ term = –0.017 m

whence EVRF2000 height $X_2 = H_{EVRF}$ = +1 * 473.0 +(–0.245) + (–0.047) + (–0.017)
= 472.69 m.

For the reverse transformaton from EVRF2000 height of 472.69 m to LN02 height:

$X_1 = H_{LN02}$ = +1 * {472.69 + [–(–0.245)] + [–(–0.047)] + [–(–0.017)]}
= 473.00 m.

2.4.3 Transformations between Geocentric Coordinate Reference Systems

The methods in this section operate in the geocentric coordinate domain. However they are often used as the middle part of a transformation of coordinates from one geographic coordinate reference system into another forming a concatenated operation of three steps:

[geographical to geocentric >> geocentric to geocentric >> geocentric to geographic]

See section 2.4.4.1 for a fuller description of these concatenated operations, and Guidance Note 7 part 1 (Use of the EPSG geodetic parameter dataset) for a more general treatment of implicit concatenated operations created by application software.

2.4.3.1 Geocentric Translations

(EPSG dataset coordinate operation method code 9603)

If we assume that the axes of the ellipsoids are parallel, that the prime meridian is Greenwich, and that there is no scale difference between the source and target coordinate reference system, then geocentric coordinate reference systems may be related to each other through three translations (colloquially known as shifts) **dX, dY, dZ** in the sense from source geocentric coordinate reference system to target geocentric coordinate reference system. They may then be applied as

$$X_t = X_s + dX$$
$$Y_t = Y_s + dY$$
$$Z_t = Z_s + dZ$$

Example:

Consider a North Sea point with coordinates derived by GPS satellite in the WGS84 geocentric coordinate reference system, with coordinates of:

X_s = 3771 793.97 m
Y_s = 140 253.34 m
Z_s = 5124 304.35 m

whose coordinates are required in terms of the ED50 coordinate reference system which takes the International 1924 ellipsoid. The three parameter geocentric translations method's parameter values from WGS84 to ED50 for this North Sea area are given as dX = +84.87m, dY = +96.49m, dZ = +116.95m.

Applying the quoted geocentric translations to these, we obtain new geocentric values now related to ED50:

X_t = 3771 793.97 + 84.87 = 3771 878.84 m
Y_t = 140 253.34 + 96.49 = 140 349.83 m
Z_t = 5124 304.35 + 116.95 = 5124 421.30 m

2.4.3.2 Helmert 7-parameter transformations

2.4.3.2.1 Position Vector 7-parameter transformation

(EPSG dataset coordinate operation method code 9606)

It is rare for the condition assumed in the geocentric translation method above – that the axes of source and target systems are exactly parallel and the two systems have an identical scale – is true. Further parameters to account for rotation and scale differences may be introduced. This is usually described as a simplified 7-parameter Helmert transformation, expressed in matrix form in what is known as the "Bursa-Wolf" formula:

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = M * \begin{pmatrix} 1 & -R_Z & +R_Y \\ +R_Z & 1 & -R_X \\ -R_Y & +R_X & 1 \end{pmatrix} * \begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} + \begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix}$$

The parameters are commonly referred to defining the transformation "from source coordinate reference system to target coordinate reference system", whereby (X_S, Y_S, Z_S) are the coordinates of the point in the source geocentric coordinate reference system and (X_T, Y_T, Z_T) are the coordinates of the point in the target geocentric coordinate reference system. But that does not define the parameters uniquely; neither is the definition of the parameters implied in the formula, as is often believed. However, the following definition, which is consistent with the "Position Vector Transformation" convention (EPSG dataset coordinate operation method code 9606), is common E&P survey practice, used by the International Association of Geodesy (IAG) and recommended by ISO 19111:

(dX, dY, dZ) :Translation vector, to be added to the point's position vector in the source coordinate reference system in order to transform from source system to target system; also: the coordinates of the origin of the source coordinate reference system in the target coordinate reference system.

(R_X, R_Y, R_Z) :Rotations to be applied to the point's vector. The sign convention is such that a positive rotation about an axis is defined as a clockwise rotation of the position vector when viewed from the origin of the Cartesian coordinate reference system in the positive direction of that axis; e.g. a positive rotation about the Z-axis only from source system to target system will result in a larger longitude value for the point in the target system. Although rotation angles may be quoted in any angular unit of measure, the formula as given here requires the angles to be provided in radians.

M : The scale correction to be made to the position vector in the source coordinate reference system in order to obtain the correct scale in the target coordinate reference system. $M = (1 + dS*10^{-6})$, where dS is the scale correction expressed in parts per million.

Example:
Transformation from WGS 72 to WGS 84 (EPSG dataset transformation code 1238). Transformation parameter values:

dX	=	0.000 m		
dY	=	0.000 m		
dZ	=	+4.5 m		
R_X	=	0.000 sec		
R_Y	=	0.000 sec		
R_Z	=	+0.554 sec	=	0.000002685868 radians
dS	=	+0.219 ppm		

Input point coordinate system: WGS 72 (Cartesian geocentric coordinates):

X_S	=	3 657 660.66 m
Y_S	=	255 768.55 m
Z_S	=	5 201 382.11 m

Application of the 7 parameter Position Vector Transformation results in:

X_T	=	3 657 660.78 m
Y_T	=	255 778.43 m
Z_T	=	5 201 387.75 m

on the WGS 84 geocentric coordinate reference system.

Reversibility
The Helmert 7-parameter transformations is an approximation formula that is valid only when the transformation parameters are small compared to the magnitude of the geocentric coordinates. Under this condition the transformation is considered to be reversible for practical purposes.

2.4.3.2.2 Coordinate Frame Rotation
(EPSG dataset coordinate operation method code 9607)

Although being common practice particularly in the European E&P industry, the Position Vector Transformation sign convention is not universally accepted. A variation on this formula is also used, particularly in the USA E&P industry. That formula is based on the same definition of translation and scale parameters, but a different definition of the rotation parameters. The associated convention is known as the "Coordinate Frame Rotation" convention.
The formula is:

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = M * \begin{pmatrix} 1 & +R_Z & -R_Y \\ -R_Z & 1 & +R_X \\ +R_Y & -R_X & 1 \end{pmatrix} * \begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} + \begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix}$$

and the parameters are defined as:

(dX, dY, dZ) : Translation vector, to be added to the point's position vector in the source coordinate reference system in order to transform from source coordinate reference system to target coordinate reference system; also: the coordinates of the origin of source coordinate reference system in the target frame.

(R_X, R_Y, R_Z) : Rotations to be applied to the coordinate reference frame. The sign convention is such that a positive rotation of the frame about an axis is defined as a clockwise rotation of the coordinate reference frame when viewed from the origin of the Cartesian coordinate reference system in the positive direction of that axis, that is a positive rotation about the Z-axis only from source coordinate reference system to target coordinate reference system will result in a smaller longitude value for the point in the target coordinate reference system. Although rotation angles may be quoted in any angular unit of measure, the formula as given here requires the angles to be provided in radians.

M : The scale factor to be applied to the position vector in the source coordinate reference system in order to obtain the correct scale of the target coordinate reference system. $M = (1+dS*10^{-6})$, where dS is the scale correction expressed in parts per million.

In the absence of rotations the two formulas are identical; the difference is solely in the rotations. The name of the second method reflects this.

Note that the same rotation that is defined as positive in the Position Vector method is consequently negative in the Coordinate Frame method and vice versa. It is therefore crucial that the convention underlying the definition of the rotation parameters is clearly understood and is communicated when exchanging transformation parameter values, so that the parameter values may be associated with the correct coordinate transformation method (algorithm).

The same example as for the Position Vector Transformation can be calculated, however the following transformation parameters have to be applied to achieve the same input and output in terms of coordinate values:

Transformation parameters Coordinate Frame Rotation convention:

dX	=	0.000 m		
dY	=	0.000 m		
dZ	=	+4.5 m		
R_X	=	–0.000 sec		
R_Y	=	–0.000 sec		
R_Z	=	–0.554 sec	=	–0.000002685868 radians
dS	=	+0.219 ppm		

Please note that only the rotation has changed sign as compared to the Position Vector Transformation. The Position Vector convention is used by IAG and recommended by ISO 19111.

The comments on reversibility of the Position Vector method apply equally to the Coordinate Frame method.

2.4.3.3 Molodensky-Badekas 10-parameter transformation

(EPSG dataset coordinate operation method code 9636)

To eliminate high correlation between the translations and rotations in the derivation of parameter values for the Helmert transformation methods discussed in the prvious section, instead of the rotations being derived about the geocentric coordinate reference system origin they may be derived at a location within the points used in the determination. Three additional parameters, the coordinates of the rotation point, are then required. The formula is:

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = M * \begin{pmatrix} 1 & +R_Z & -R_Y \\ -R_Z & 1 & +R_X \\ +R_Y & -R_X & 1 \end{pmatrix} * \begin{pmatrix} X_S & - & X_P \\ Y_S & - & Y_P \\ Z_S & - & Z_P \end{pmatrix} + \begin{pmatrix} X_P \\ Y_P \\ Z_P \end{pmatrix} + \begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix}$$

and the parameters are defined as:

(dX, dY, dZ) : Translation vector, to be added to the point's position vector in the source coordinate system in order to transform from source coordinate reference system to target coordinate reference system; also: the coordinates of the origin of source coordinate reference system in the target frame.

(R_X, R_Y, R_Z) : Rotations to be applied to the coordinate reference frame. The sign convention is such that a positive rotation of the frame about an axis is defined as a clockwise rotation of the coordinate reference frame when viewed from the origin of the Cartesian coordinate system in the positive direction of that axis, that is a positive rotation about the Z-axis only from source coordinate reference system to target coordinate reference system will result in a smaller longitude value for the point in the target coordinate reference system. Although rotation angles may be quoted in any angular unit of measure, the formula as given here requires the angles to be provided in radians.

(X_P, Y_P, Z_P) : Coordinates of the point about which the coordinate reference frame is rotated, given in the source Cartesian coordinate reference system.

M : The scale factor to be applied to the position vector in the source coordinate reference system in order to obtain the correct scale of the target coordinate reference system. $M = (1+dS*10^{-6})$, where dS is the scale correction expressed in parts per million.

The [Helmert 7-parameter] Coordinate Frame Rotation method discussed in the previous section is a specific case of the Molodensky-Badekas 10-parameter transformation in which the evaluation point is the origin of the geocentric coordinate system, at which geocentric coordinate values are zero.

Example

See section 2.4.4.1 below for an example.

Reversibility
The Molodensky-Badekas 10-parameter transformation strictly speaking is not reversible, i.e. in principle the same parameter values cannot be used to execute the reverse transformation. This is because the evaluation point coordinates are in the forward direction source coordinate reference system and the rotations have been derived about this point. They should not be applied about the point having the same coordinate values in the target coordinate reference system, as is required for the reverse transformation. However, in practical application there are exceptions when applied to the approximation of small differences in the geometry of a set of points in two different coordinate reference systems. The typical vector difference in coordinate values is in the order of $6*10^1$ to $6*10^2$ metres, whereas the evaluation point on or near the surface of the earth is $6.3*10^6$ metres from the origin of the coordinate systems at the Earth's centre. This difference of four or five orders of magnitude allows the transformation in practice to be considered reversible. Note that in the reverse transformation, only the signs of the translation and rotation parameter values and scale are reversed; the coordinates of the evaluation point remain unchanged.

2.4.4 Transformations between Geographic Coordinate Reference Systems

2.4.4.1 Transformations using geocentric methods

Transformation of coordinates from one geographic coordinate reference system into another is often carried out as a concatenation of three operations:

geographical (3D) to geocentric + geocentric to geocentric + geocentric to geographic (3D)

The middle step of the concatenated transformation, from geocentric to geocentric, may be through any of the methods described in section 2.4.3 above: geocentric translations, 7-parameter Helmert transformation or 10-parameter Molodensky-Badekas transformation. The first and last steps of the concatenated transformation (geographic 3D to/from geocentric) are described in section 2.2.1 above.

If involving geographic 2D coordinates, the techniques described in section 2.2.2 above (geographic 3D to/from 2D) may also be used as additional steps at each end of the concatenation.

Example
Transformation from La Canoa to REGVEN (EPSG dataset transformation code 1771). The 10 Molodensky-Badekas transformation parameter values are:

dX	=	-270.933 m		
dY	=	+115.599 m		
dZ	=	-360.226 m		
R_X	=	-5.266 sec	=	-0.000025530288 radians
R_Y	=	-1.238 sec	=	-0.000006001993 radians
R_Z	=	+2.381 sec	=	+0.000011543414 radians
dS	=	-5.109 ppm		
Ordinate 1 of evaluation point	=	2464351.59 m		
Ordinate 2 of evaluation point	=	-5783466.61 m		
Ordinate 3 of evaluation point	=	974809.81 m		

Ellipsoid Parameters for the source and target coordinate reference systems are are:

CRS name	Ellipsoid name	Semi-major axis (a)	Inverse flattening (1/f)
La Canoa	International 1924	6378388.0 metres	1/f = 297.0
REGVEN	WGS 84	6378137.0 metres	1/f = 298.2572236

Input point coordinate system: La Canoa (geographic 2D)

Latitude φ_S	=	9°35'00.386"N
Longitude λ_S	=	66°04'48.091"W

Using the technique described in section 2.2.1 above, this is taken to be geographic 3D with an assumed ellipsoidal height h_S = 201.46 m

Using the geographic (3D) to geocentric conversion method given in section 2.2.1, these three coordinates convert to Cartesian geocentric coordinates:

X_S = 2 550 408.96 m
Y_S = -5 749 912.26 m
Z_S = 1 054 891.11 m

Application of the 10 parameter Molodensky-Badekas Transformation (section 2.4.3.3) results in:

X_T = 2 550 138.46 m
Y_T = -5 749 799.87 m
Z_T = 1 054 530.82 m

on the REGVEN geocentric coordinate reference system (CRS code 4962)

Using the reverse formulas for the geographic/geocentric conversion method given in section 2.2.1 on the REGVEN geographic 3D coordinate reference system (CRS code 4963) this converts into:

Latitude φ_T = 9°34'49.001"N
Longitude λ_T = 66°04'54.705"W
Ellipsoidal height h_T = 180.51 m

Because the source coordinates were 2D, the target system ellipsoidal height is ignored (see section 2.2.2 above) and the results treated as a geographic 2D coordinate reference system (CRS code 4189):

Latitude φ_T = 9°34'49.001"N
Longitude λ_T = 66°04'54.705"W

2.4.4.1.1 <u>France geocentric interpolation</u>
(EPSG dataset coordinate operation method code 9655)

In France the national mapping agency (IGN) have promolgated a transformation between the classical geographic 2D coordinate reference system NTF and the modern 3-dimensional system RGF93 which uses geocentric translations interpolated from a grid file. The method is described in IGN document NTG-88. In summary:

- The grid file nodes are given in RGF93 geographic 2D coordinates.
- Within the grid file the sense of the parameter values is *from* NTF *to* RGF93.

For NTF to RGF93 transformations an iteration to obtain coordinates in the appropriate system for interpolation within the grid is required. The steps are:

- Convert NTF geographic 2D coordinates to geographic 3D by assuming a height and then to NTF geocentric coordinates.
- Transform NTF geocentric coordinates to approximate RGF93 coordinates using an average value for all France (EPSG dataset coordinate operation code 1651):

  $X_{NTF} = X_{RGF93'}$ -168 m
  $Y_{NTF} = Y_{RGF93'}$ -60 m
  $Z_{NTF} = Z_{RGF93'}$ +320 m
- Convert the approximate RGF93 geocentric coordinates to approximate RGF93 geographic coordinates.
- Using the approximate RGF93 geographic coordinates, interpolate within the grid file to obtain the three geocentric translations (dX, dY, dZ) applicable at the point.
- Apply these geocentric translations to the NTF geocentric coordinates to obtain RGF93 geocentric coordinates:

- Transform RGF93 geocentric coordinates to NTF geocentric coordinates, taking account of the sense of the parameter values.

$$X_{RGF93} = X_{NTF} + dX$$
$$Y_{RGF93} = Y_{NTF} + dY$$
$$Z_{RGF93} = Z_{NTF} + dZ$$

- Convert RGF93 geocentric coordinates to RGF93 geographic 3D coordinates. Because the original input NTF coordinates were geographic 2D, the RGF93 ellipsoidal height is meaningless so it is dropped to give RGF93 geographic 3D coordinates.

For RGF93 to NTF transformations the steps are:

- Using the RGF93 geographic coordinates, interpolate within the grid file to obtain the three geocentric translations (dX, dY, dZ) applicable at the point.
- Convert RGF93 geographic coordinates to RGF93 geocentric coordinates.
- Transform RGF93 geocentric coordinates to NTF geocentric coordinates, taking into account the sense of the parameter values:

$$X_{NTF} = X_{RGF93} + (-\,dX)$$
$$Y_{NTF} = Y_{RGF93} + (-\,dY)$$
$$Z_{NTF} = Z_{RGF93} + (-\,dZ)$$

- Convert NTF geocentric coordinates to geographic 3D coordinates.
- Drop the ellipsoid height to give NTF geographic 2D coordinates.

2.4.4.2 <u>Abridged Molodensky transformation</u>

(EPSG dataset coordinate operation method code 9605)

As an alternative to the computation of the new latitude, longitude and ellipsoid height by concatenation of three operations (geographical to geocentric + geocentric to geocentric + geocentric to geographic), the changes in these coordinates may be derived directly as geographical coordinate offsets through formulas derived by Molodensky (EPSG dataset coordinate operation method code 9604, not detailed in this Guidance Note). Abridged versions of these formulas, which quite satisfactory for three-parameter geocentric transformations, are as follows:

$$\varphi_t = \varphi_s + d\varphi$$
$$\lambda_t = \lambda_s + d\lambda$$
$$h_t = h_s + dh$$

where

$$d\varphi\,'' = (-\,dX \sin\varphi_s \cos\lambda_s - dY \sin\varphi_s \sin\lambda_s + dZ \cos\varphi_s + [a\,df + f\,da] \sin 2\varphi_s) / (\rho_s \sin 1'')$$
$$d\lambda\,'' = (-\,dX \sin\lambda_s + dY \cos\lambda_s) / (\nu_s \cos\varphi_s \sin 1'')$$
$$dh = dX \cos\varphi_s \cos\lambda_s + dY \cos\varphi_s \sin\lambda_s + dZ \sin\varphi_s + (a\,df + f\,da) \sin^2\varphi_s - da$$

and where dX, dY and dZ are the geocentric translation parameters, ρ_s and ν_s are the meridian and prime vertical radii of curvature at the given latitude φ_s on the first ellipsoid, da is the difference in the semi-major axes of the target and source ellipsoids and df is the difference in the flattening of the two ellipsoids:

$$\rho_s = a_s (1 - e_s^2) / (1 - e_s^2 \sin^2\varphi_s)^{3/2}$$
$$\nu_s = a_s / (1 - e_s^2 \sin^2\varphi_s)^{1/2}$$
$$da = a_t - a_s$$
$$df = f_t - f_s = 1/(1/f_t) - 1/(1/f_s).$$

The formulas for $d\varphi$ and $d\lambda$ indicate changes in φ and λ in arc-seconds.

Example:

For a North Sea point with coordinates derived by GPS satellite in the WGS84 geographical coordinate reference system, with coordinates of:

latitude φ_s = 53°48'33.82"N,
longitude λ_s = 2°07'46.38"E,
and ellipsoidal height h_s = 73.0m,

whose coordinates are required in terms of the ED50 geographical coordinate reference system which takes the International 1924 ellipsoid.

The three geocentric translations parameter values *from* WGS 84 *to* ED50 for this North Sea area are given as dX = +84.87m, dY = +96.49m, dZ = +116.95m.
Ellipsoid Parameters are:

WGS 84	a = 6378137.0 metres	1/f = 298.2572236
International 1924	a = 6378388.0 metres	1/f = 297.0

Then

da = 6378388 – 6378137 = 251
df = 0.003367003 – 0.003352811 = 1.41927E-05

whence

$$d\varphi = (-68.448-2.894+69.056+87.079)/30.917 = 2.743''$$
$$d\lambda = (-3.154+96.423)/18.299 = 5.097''$$
$$dh = 50.079+2.117+94.385+59.510-251.000 = -44.909 \text{ m}$$

ED50 values on the International 1924 ellipsoid are then:

latitude φ_t = 53°48'36.563"N
longitude λ_t = 2°07'51.477"E
and ellipsoidal height h_t = 28.091 m

Because ED50 is a geographical 2D coordinate reference system the height is dropped to give:

latitude φ_t = 53°48'36.56"N
longitude λ_t = 2°07'51.48"E

For comparison, better values computed through the concatenation of the three operations (geographical to geocentric + geocentric to geocentric + geocentric to geographic) are:

latitude φ_t = 53°48'36.565"N
longitude λ_t = 2°07'51.477"E
and ellipsoidal height h_t = 28.02 m

2.4.4.3 Geographic Offsets

This is the simplest of transformations between two geographic coordinate reference systems, but is normally used only for purposes where low accuracy can be tolerated. It is generally used for transformations in two dimensions, latitude and longitude, where:

$$\varphi_t = \varphi_s + d\varphi$$
$$\lambda_t = \lambda_s + d\lambda$$

(EPSG dataset coordinate operation method code 9619).

In very rare circumstances, a transformation in three dimensions additionally including ellipsoidal height may be encountered:

$$\varphi_t = \varphi_s + d\varphi$$

$$\lambda_t = \lambda_s + d\lambda$$
$$h_t = h_s + dh$$

(EPSG coordinate operation method code 9660)

This should not be confused with the Geographic2D with Height Offsets method used in Japan, where the height difference is between the ellipsoidal height component of a 3D geographic coordinate reference system and a gravity-related height system. This is discussed in section 2.4.5 below.

Example:
A position with coordinates of 38°08'36.565"N, 23°48'16.235"E referenced to the old Greek geographic 2D coordinate reference system (EPSG datset CRS code 4120) is to be transformed to the newer GGRS87 system (EPSG dataset CRS code 4121). Transformation parameters from Greek to GGRS87 are:

$$d\varphi = -5.86''$$
$$d\lambda = +0.28''$$

Then φ_{GGRS87} = 38°08'36.565"N + (-5.86") = 38°08'30.705"N
and λ_{GGRS87} = 23°48'16.235"E + 0.28" = 23°48'16.515"E

For the reverse transformation for the same point,
φ_{GREEK} = 38°08'30.705"N + 5.86" = 38°08'36.565"N
λ_{GREEK} = 23°48'16.515"E + (-0.28") = 23°48'16.235"E

2.4.4.4 Geographic Offset by Interpolation of Gridded Data

The relationship between some geographical 2D coordinate reference systems is available through gridded data sets of latitude and longitude offsets. This family of methods includes:

NADCON (EPSG dataset coordinate operation method code 9613) which is used by the US National Geodetic Survey for transformation between US systems,

NTv2 (EPSG dataset coordinate operation method code 9615) which originated in the national mapping agency of Canada and was subsequently adopted in Australia and New Zealand, and

OSTN (EPSG dataset coordinate operation method code 9633) used in Great Britain.

The offsets at a point are derived by interpolation within the gridded data. In some methods, separate grid files are given for latitude and longitude offsets whilst in other methods the offsets for both latitude and longitude are given within a single grid file. The EPSG dataset differentiates methods by the format of the gridded data file(s). The grid file format is given in documentation available from the information source. Although the authors of some data sets suggest a particular interpolation method within the grid(s), generally the density of grid nodes should be such that any reasonable grid interpolation method will give the same offset value. Bi-linear interpolation is the most usual grid interpolation mechanism. The interpolated value of the offset A is then added to the source CRS coordinate value to give the coordinates in the target CRS.

Reversibility
The coordinate reference system for the coordinates of the grid nodes will be given either in the file itself or in accompanying documentation. This will normally be the source coordinate reference system for the forward transformation. Then in forward transformations the offset is obtained through straightforward interpolation of the grid file. But for the reverse transformation the first grid interpolation entry will be the value of the point in the second coordinate reference system, the offsets are interpolated and applied *with sign reversed*, and the result used in further iterations of interpolation and application of offset until the difference between results from successive iterations is insignificant.

2.4.5 Geoid and Height Correction Models

2.4.5.1 Geographic3D to GravityRelatedHeight

Although superficially involving a change of dimension from three to one, this transformation method is actually one-dimensional. The transformation applies an offset to the ellipsoidal height component of a geographic 3D coordinate reference system with the result being a gravity-related height in a vertical coordinate reference system. However the ellipsoidal height component of a geographic 3D coordinate reference system cannot exist without the horizontal components, i.e. it cannot exist as a one-dimensional coordinate reference system.

Geodetic science distinguishes between geoid-ellipsoid separation models and height correction models. Geoid separation models give the height difference between the ellipsoid and the geoid surfaces. Height correction models give height difference between ellipsoidal a particular vertical datum surface. Because a vertical datum is a realisation of the geoid and includes measurement errors and various constraints, a vertical datum surface will not exactly coincide with the geoid surface. The mathematics of the application of these models is identical and for the purposes of the EPSG dataset they are considered to be one method.

The correction value ζ [6] is interpolated from a grid of height differences and the interpolation requires the latitude and longitude components of the geographic 3D coordinate reference system as arguments.

If **h** is the ellipsoidal height (height of point above the ellipsoid, positive if up) in the geographic 3D CRS and **H** is the gravity-related height in a vertical CRS, then

$$H = h - \zeta$$

Note that unlike the general convention adopted for offsets described in 2.4.1, geoid separation and height correction models conventionally use the true mathematical convention for sign.

The EPSG dataset differentiates between the formats of the gridded height files and distinguishes separate coordinate operation methods for each file format. The coordinate operation method may also define the interpolation technique to be used. However the density of grid nodes is usually sufficient for any reasonable interpolation technique to be used, with bi-linear interpolation usually being applied.

Reversibility

The reverse transformation, from gravity-related height in the vertical coordinate reference system to the ellipsoidal height component of the geographic3D coordinate reference system, requires that a horizontal position be associated with the gravity-related height. This is indeterminate unless a compound coordinate reference system is involved (see the Geographic3D to Geographic2D+GravityRelatedHeight method described below). Geographic3D to GravityRelatedHeight methods therefore are not reversible.

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[6] Geodetic science recognises several types of gravity-related height, differentiated by assumptions made about the gravitational field. A discussion of these types is beyond the scope of this document. In this document the symbol ζ is used to indicate the correction to be applied to the ellipsoid height.

2.4.5.2 Geographic3D to Geographic2D+GravityRelatedHeight

This method transforms coordinates between a geographic 3D coordinate reference system and a compound coordinate reference system consisting of separate geographic 2D and vertical coordinate reference systems. Separate operations are made between the horizontal and vertical components. In its simplest form it combines a Geographic 3D to 2D conversion and a Geographic3D to GravityRelatedHeight transformation (see sections 2.2.2 and 2.4.5.1 above). However, complexities arise (a) for the forward transformation if the source 3D and target 2D geographic coordinate reference systems are based on different geodetic datums, or (b) in the reverse transformation of height from compound to geographic 3D.

Horizontal component
If the horizontal component of the compound coordinate reference system and the geographic 3D coordinate reference system are based on the same geodetic datum, this operation is simply the Geographic 3D to 2D conversion described in section 2.2.2 above except that for the reverse case (2D to 3D) no assumption is required for the ellipsoidal height as it will come from the operation for the vertical part.

If the horizontal component of the compound coordinate reference system and the geographic 3D coordinate reference system are based on different geodetic datums then any of the geographic to geographic transformations discussed in section 2.4.4 above, including those using geocentric methods (sections 2.4.3 and 2.4.4.1), may be used.

Vertical component
The forward transformation from geographic 3D to vertical component of the compound system uses the Geographic3D to GravityRelatedHeight method described in section 2.4.5.1 above. Then:

$$H = h - \zeta$$

where, as before, **h** is the ellipsoidal height (height of point above the ellipsoid, positive if up) in the geographic 3D CRS, **H** is the gravity-related height in the vertical CRS part of the compound CRS and ζ is the correction from ellipsoidal height to gravity-related height from the gridded data.

The reverse transformation, from vertical component of the compound system to geographic 3D system, requires interpolation within the grid of height differences. However the latitude and longitude arguments for this interpolation must be in the geographic 3D coordinate reference system, as the nodes for the gridded data will be in this system. Therefore the reverse operation on the horizontal component of the compound system must be executed before the reverse vertical transformation can be made. Then:

$$h = H - -\zeta$$

2.4.5.3 Geographic2D with Height Offsets

(EPSG dataset coordinate operation method code 9618)

This method used in Japan is a simplified form of the general Geographic3D to Geographic2D+GravityRelatedHeight method described above. It combines the geographical 2D offset method described in section 2.4.4.3 above with an ellipsoidal height to gravity-related height value A applied as a vertical offset.

$$\varphi_{WGS84} = \varphi_{Tokyo} + d\varphi$$
$$\lambda_{WGS84} = \lambda_{Tokyo} + d\lambda$$
$$h_{WGS84} = H_{JSLD} + A$$